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MASTER'S THESIS

Local Sampling Analysis for Quadratic Embeddings of Riemannian Manifolds

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Abstract

In this work we do a theoretical analysis of the local sampling conditions for points lying on a quadratic embedding of a Riemannian manifold in a Euclidean space. The embedding is assumed to be quadratic at a reference point P. Our analysis is based on the following criteria: (i) Local reconstruction error (ii) Local tangent space estimation accuracy. In the local reconstruction error analysis we describe sampling conditions in the neighbourhood of P such that the average reconstruction error of the samples after orthogonal projection on the local tangent space, satisfies a given upper bound. We derive a lower bound on the number of neighbouring samples which probabilistically guarantees that a predefined local reconstruction error criterion will be satisfied. In local tangent space estimation analysis, we analyze the locally estimated linear subspace, which is optimal in the least squares sense and passes through P. The tangent space at P is estimated using the samples lying in its neighbourhood. Sampling conditions for the neighbourhood points are derived so that the "angle" [2] between the estimated tangent space and the original tangent space at P is upper bounded. We again consider both probabilistic and non-probabilistic sampling conditions for this criterion. We derive a lower bound on the number of neighbouring samples which probabilistically guarantees an upper bound on the "angle" between the estimated tangent space and the original tangent space.

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Chapter 1

Introduction

1.1 Background

A set of high dimensional data that can be locally mapped to a lower dimensional Euclidean space constitutes a manifold in the high dimensional space. An example of such a low dimensional manifold embedded in a high dimensional space is a set of images representing the same 3D object, captured under different camera views. Thus, though the dimensionality of each image is high (number of pixels) the intrinsic dimensionality of the set of images is low (intrinsic and extrinsic parameters of the camera etc). The recovery of the manifold structure underlying a set of data has been a popular research topic of the recent years. In manifold learning, a representation of the data with reduced dimensionality is searched, which also helps in understanding the structure of the signal. It is possible to retrieve the manifold structure of data in various ways, such as by describing a global parameterization based on geodesic distances as in ISOMAP [4], or via locally linear representations as in LLE [5] or Hessian Eigenmaps [6].
Given a set of input data, the ISOMAP algorithm [4] determines the nearest neighbors in the data, constructs a graph with respect to nearest neighbors, and then obtains an approximation of the geodesic distance on the manifold by summing the Euclidean distances between neighboring nodes. Once the geodesic distances between all pairs of samples are estimated, then the algorithm computes a mapping of the initial data into a low dimensional Euclidean space such that the geodesic distances on the manifold are proportional to the Euclidean distances in the embedded space. However, one main assumption of the algorithm is that the geodesic curves on the manifold can be well-approximated with the piecewise linear curves constructed using data samples. The accuracy of this approximation depends on the *sampling*

rate of the data on the manifold, i.e., as more data samples on the manifold are available to the algorithm, the estimation of geodesic distances is more accurate and the algorithm performs better. An analysis of the performance of the ISOMAP algorithm with respect to the sampling conditions on the manifold is made in [7].
Manifold learning algorithms such as LLE [5] or Hessian Eigenmaps [6] have the same purpose of obtaining a parameterization of data, however, the working principle of these algorithms is slightly different than ISOMAP. Instead of computing geodesic distances, the LLE algorithm considers the locally linear structure of the manifold, and tries to represent each data sample on the manifold by a weighted combination of its nearest neighbors. The Hessian Eigenmaps algorithm is similar to LLE in the sense that it is based on locally linear approximations of the manifold. However, it has been seen to be more robust than LLE as it also takes into account more detailed geometric characteristics of the manifold. It is clear that the performance of such algorithms is also strictly dependent on parameters such as the *sampling rate* of data and the geometric properties of the manifold such as curvature.

In this thesis we do a theoretical analysis of the local sampling conditions for points lying on a given quadratic embedding of a Riemannian manifold in a Euclidean space. The embedding is assumed to be quadratic at a reference point P. The motivation behind considering quadratic embeddings is to get an intuition of local sampling conditions for the more general case of C^2 embeddings. In particular we get an understanding of smooth (C^∞) Riemannian manifolds which locally have a good quadratic approximation at each point. Furthermore, the quadratic embedding assumption enables us to analyze the local geometry in terms of the principal curvatures of the manifold at each point. Our analysis is based on the following criteria: (i) Local reconstruction error (ii) Local tangent space estimation accuracy. In the local reconstruction error analysis we describe sampling conditions in the neighbourhood of P such that the average reconstruction error of the points after orthogonal projection on the tangent space at P, satisfies a given upper bound. We consider both probabilistic and non-probabilistic sampling conditions for our analysis. In local tangent space estimation analysis, we analyze the locally estimated linear subspace, which is optimal in the least squares sense and passes through P. The tangent space is estimated using the points lying in the neighbourhood of P. Sampling conditions for the neighbourhood points are derived so that the "angle" [2] between the estimated tangent space and the original tangent space at P is upper bounded. We again consider both probabilistic and non-probabilistic sampling conditions for this criterion.

1.2 Problem Statement

Consider an m-dimensional manifold S embedded in $\mathbb{R}^n$, where $n \geq m+1$ and the manifold is described by $n-m$ quadratic functions at a reference point P. At P the embedding is given by the graphs of the functions

$$f_i : T_P S \to T_P S^{\perp} \ ; \ \forall \ i = 1, \ldots, n-m,$$

where $T_P S$ denotes the tangent space at P, $T_P S^{\perp}$ denotes the orthogonal complement of the tangent space [1] and f_i is a quadratic function $\forall \ i = 1, \ldots, n-m$. This implies that $\bigtriangledown_f = \bar{0}$ at P. Furthermore any point on S can be written as: $[x_1 \ldots x_m \ f_1(\bar{x}) \ldots f_{n-m}(\bar{x})]$ where $\bar{x} = [x_1 \ldots x_m]$ denotes the coordinates of the point in $T_P S$.
Consider a set of discrete points $\mathcal{A}$ lying on S. Let $\mathcal{N}_\varepsilon(P)$ denote the set of samples lying in the ε-neighbourhood of P, for some $\varepsilon > 0$, where

$$\mathcal{N}_\varepsilon(P) \ = \ \{M \in \mathcal{A} : \| \ M - P \ \| \leq \varepsilon\}, \quad |\mathcal{N}_\varepsilon(P)| = K.$$

Here $\| \cdot \|$ denotes the l_2 norm. We assume that $K \geq 1$. Given the above, we would now like to solve the following problems.

(a) Describe sufficient conditions on the points in $\mathcal{N}_\varepsilon(P)$ such that the average reconstruction error of the points in $\mathcal{N}_\varepsilon(P)$, after projection on $T_P S$ is less than a specified value $\frac{\gamma^2}{4}$, where $\gamma \geq 0$. The reconstruction error of a point $M \in \mathcal{N}_\varepsilon(P)$ is defined as $\| \ M - \hat{M} \ \|^2$ where $\hat{M}$ is the orthogonal projection of M on $T_P S$.

(b) Say we find the m-dimensional linear subspace $\hat{T}_P S$ passing through P, that is optimal in the quadratic error sense. $\hat{T}_P S$ is estimated by performing a local principal component analysis (PCA) using the points in $\mathcal{N}_\varepsilon(P)$. Describe the conditions on the points in $\mathcal{N}_\varepsilon(P)$ so that $|\theta| < |\phi| < \frac{\pi}{2}$. Here θ is the "angle"(as defined in [2]) between $\hat{T}_P S$ and $T_P S$. $|\phi|$ denotes the angle bound.

1.3 Thesis Outline

The rest of the thesis is organized as follows. In Chapter 2 we analyze local sampling conditions for 2-dimensional quadratic surfaces embedded in $\mathbb{R}^3$. In particular Sections 2.1-2.4 contain the sampling analysis for local reconstruction error criterion and Sections 2.5-2.7 contain the sampling analysis for local tangent space estimation analysis. In Chapter 3 we analyze local sampling conditions for quadratic embeddings of m-dimensional Riemannian manifolds in $\mathbb{R}^n$, where $n \geq m+1$. In particular Sections 3.1-3.2 contain the sampling analysis for local reconstruction error criterion and Sections 3.3-3.4 contain the sampling analysis for local tangent space estimation analysis.

Chapter 2

2-dimensional Quadratic Surfaces in $\mathbb{R}^3$

In this chapter we consider 2-dimensional quadratic surfaces embedded in $\mathbb{R}^3$. The embedding is assumed to be quadratic at a reference point P. We derive sampling conditions for points lying in $\mathcal{N}_\varepsilon(P)$ such that they guarantee the specified performance criterion. In Section 2.1 we consider the trivial case where there is a single point in $\mathcal{N}_\varepsilon(P)$, and derive bounds on its norm such that a reconstruction error criterion is satisfied. In Section 2.2 we consider the case where $\mathcal{N}_\varepsilon(P)$ contains more than one point. We partition $\mathcal{N}_\varepsilon(P)$ into two disjoint subsets namely S_1 and S_2, consisting of points from the local "high" curvature and "low" curvature regions respectively. We derive bounds on the norms of points lying in these sets and also precisely describe the geometry of the corresponding regions in $T_P S$. In Section 2.3 we consider a framework in which the points are sampled uniformly at random from the regions corresponding to S_1 and S_2 in $T_P S$. We derive a condition for the scaling factors of the bounds of the norms for points in S_1 and S_2 which were derived in Section 2.2. We show that if the scaling factors satisfy this condition and if the number of samples K satisfies a lower bound then it guarantees an upper bound on the probability of the event where the empirical average reconstruction error of the points in $\mathcal{N}_\varepsilon(P)$ exceeds the reconstruction error criterion. Section 2.4 contains simulation results for local reconstruction error analysis using synthetic surfaces.

In Section 2.5 we derive conditions for points in $\mathcal{N}_\varepsilon(P)$ so that the "angle" between the $T_P S$ and $\hat{T}_P S$ is upper bounded. Section 2.6 contains results for the case where the points are sampled uniformly at random. We derive bounds on the norm of the points and also a lower bound on the number of samples K so that if both these conditions are met, then they guarantee probabilistically an upper bound on the "angle" between $T_P S$ and $\hat{T}_P S$. Finally in Section 2.7 we present simulation results for the conditions derived in Sections 2.5-2.6 with synthetic surfaces.

Let $M = [x_1\ x_2\ f(x_1,x_2)]$ be a point in $\mathcal{N}_\varepsilon(P)$. Let $\bar{x}_M = [x_1\ x_2]$ which is orthogonal projection of M on T_PS. As the surface S is quadratic we have the following.

$$\begin{aligned}
f(\bar{x}_M) &= f(0,0) + \bigtriangledown_f(0,0)^T\bar{x}_M + \frac{1}{2}\bar{x}_M^T H_f(0,0)\bar{x}_M \\
&= 0 + \bar{0}^T\bar{x}_M + \frac{1}{2}\bar{x}_M^T V\Lambda V^T\bar{x}_M \\
&= \frac{1}{2}\left(< \bar{x}_M, \bar{v_1} >^2 \mathcal{K}_{f,max} + < \bar{x}_M, \bar{v_2} >^2 \mathcal{K}_{f,min}\right) \\
&= \frac{1}{2} \parallel \bar{x}_M \parallel^2 \left(\frac{< \bar{x}_M, \bar{v_1} >^2}{\parallel \bar{x}_M \parallel^2}\mathcal{K}_{f,max} + \frac{< \bar{x}_M, \bar{v_2} >^2}{\parallel \bar{x}_M \parallel^2}\mathcal{K}_{f,min}\right) \\
&= \frac{1}{2} \parallel \bar{x}_M \parallel^2 \left(r_{1,M}\mathcal{K}_{f,max} + r_{2,M}\mathcal{K}_{f,min}\right) \\
&= \frac{1}{2} \parallel \bar{x}_M \parallel^2 \mathcal{K}_{f,M}
\end{aligned}$$

where

$$V = \begin{bmatrix}\bar{v_1} & \bar{v_2}\end{bmatrix}, \qquad \Lambda = \begin{bmatrix}\mathcal{K}_{f,max} & 0 \\ 0 & \mathcal{K}_{f,min}\end{bmatrix}$$

are respectively the eigenvector and eigenvalue matrices of $H_f(0,0)$. $\bar{v}_1$ and $\bar{v}_2$ geometrically represent the principal curvature directions of the surface at P. $\mathcal{K}_{f,max}$ (resp. $\mathcal{K}_{f,min}$) corresponds to the principal curvature along the direction $\bar{v}_1$ (resp. $\bar{v}_2$). Furthermore, $r_{1,M} = \dfrac{< \bar{x}_M, \bar{v_1} >^2}{\parallel \bar{x}_M \parallel^2}$, $r_{2,M} = \dfrac{< \bar{x}_M, \bar{v_2} >^2}{\parallel \bar{x}_M \parallel^2}$ so that, $r_{1,M} + r_{2,M} = 1$ and $r_{1,M}, r_{2,M} \geq 0$. Geometrically, $\mathcal{K}_{f,M} = r_{1,M}\mathcal{K}_{f,max} + r_{2,M}\mathcal{K}_{f,min}$ represents the curvature at P of the geodesic curve from P to M. Observe that

$$\text{if} \quad \mathcal{K}_{f,min}\mathcal{K}_{f,max} \geq 0 \text{ then } |\mathcal{K}_{f,M}| \in [|\mathcal{K}_{f,min}|, |\mathcal{K}_{f,max}|], \tag{2.1}$$

$$\text{and if} \quad \mathcal{K}_{f,min}\mathcal{K}_{f,max} \leq 0 \text{ then } |\mathcal{K}_{f,M}| \in [0, |\mathcal{K}_{f,max}|]. \tag{2.2}$$

So, $|\mathcal{K}_{f,M}| \in [|\mathcal{K}_{f,low}|, |\mathcal{K}_{f,max}|]$, where $\mathcal{K}_{f,low} = |\mathcal{K}_{f,min}|$ if (2.1) is satisfied and $\mathcal{K}_{f,low} = 0$ if (2.2) is satisfied. We assume w.l.o.g that $|\mathcal{K}_{f,max}| \geq |\mathcal{K}_{f,min}|$. Observe that if $|\mathcal{K}_{f,max}| = 0$ then S is nothing but T_PS itself. So we assume for the rest of the analysis that $|\mathcal{K}_{f,max}| > 0$.

2.1 Performance analysis of local reconstruction error for single neighbouring sample

Let $E_M^2 \;=\; f^2(\bar{x}_M)$ denote the reconstruction error for M.

$$\text{Hence,} \quad E_M^2 \leq \frac{\gamma^2}{4} \;\Leftrightarrow |f(\bar{x}_M)| \leq \frac{\gamma}{2} \;\Leftrightarrow \parallel \bar{x}_M \parallel^2 |\mathcal{K}_{f,M}| \leq \gamma \tag{2.3}$$

$$\begin{aligned}
\text{But} \quad & \parallel \bar{x}_M \parallel^2 |\mathcal{K}_{f,M}| \;\leq\; \parallel \bar{x}_M \parallel^2 |\mathcal{K}_{f,max}|. \\
\text{Thus, if} \quad & \parallel \bar{x}_M \parallel^2 |\mathcal{K}_{f,max}| \leq \gamma \\
& \quad \text{then} \quad \parallel \bar{x}_M \parallel \leq \sqrt{\frac{\gamma}{|\mathcal{K}_{f,max}|}}.
\end{aligned} \tag{2.4}$$

Any point M which satisfies (2.4) will also satisfy (2.3). We now proceed to find the bound on the norm of a point in the ambient space, which ensures that (2.4) is satisfied. Let ε_M denote the norm of M in the ambient space.

$$\begin{aligned}
&\text{Say, } \varepsilon_M^2 \leq C^2 \\
&\Rightarrow \frac{1}{4} \parallel \bar{x}_M \parallel^4 |\mathcal{K}_{f,M}|^2 + \parallel \bar{x}_M \parallel^2 \leq C^2 \\
&\Rightarrow \parallel \bar{x}_M \parallel^4 |\mathcal{K}_{f,M}|^2 + 4 \parallel \bar{x}_M \parallel^2 - 4C^2 \leq 0 \\
&\Rightarrow \parallel \bar{x}_M \parallel^2 \leq \frac{2C^2}{1+\sqrt{1+|\mathcal{K}_{f,M}|^2 C^2}}
\end{aligned}$$

Observe that since $\parallel \bar{x}_M \parallel \leq \sqrt{\frac{\gamma}{|\mathcal{K}_{f,max}|}}$, we need to find a condition on C such that the following holds

$$\varepsilon_M^2 \;\leq C^2 \;\Rightarrow \parallel \bar{x}_M \parallel \leq \sqrt{\frac{\gamma}{|\mathcal{K}_{f,max}|}}.$$

Observe that

$$\begin{aligned}
\parallel \bar{x}_M \parallel^2 &\leq \frac{2C^2}{1+\sqrt{1+|\mathcal{K}_{f,M}|^2 C^2}} \\
&\leq \frac{2C^2}{1+\sqrt{1+|\mathcal{K}_{f,low}|^2 C^2}}.
\end{aligned}$$

$$\text{So if } \frac{2C^2}{1+\sqrt{1+|\mathcal{K}_{f,low}|^2C^2}} \leq \frac{\gamma}{|\mathcal{K}_{f,max}|}$$
$$\Rightarrow 2|\mathcal{K}_{f,max}|C^2 \leq \gamma + \gamma\sqrt{1+|\mathcal{K}_{f,low}|^2C^2}$$
$$\Rightarrow (2|\mathcal{K}_{f,max}|C^2 - \gamma)^2 \leq \gamma^2(1+|\mathcal{K}_{f,low}|^2C^2)$$
$$\Rightarrow C^2 \leq \left(\frac{\gamma}{|\mathcal{K}_{f,max}|} + \frac{\gamma^2|\mathcal{K}_{f,low}|^2}{4|\mathcal{K}_{f,max}|^2}\right)$$

$$\text{Thus, } \varepsilon_M \ \leq \ \sqrt{\left(\frac{\gamma}{|\mathcal{K}_{f,max}|} + \frac{\gamma^2|\mathcal{K}_{f,low}|^2}{4|\mathcal{K}_{f,max}|^2}\right)} \tag{2.5}$$

Discussion: Before proceeding, it's worthwhile to interpret the conditions derived for the single point case. Observe that in practice one would not a priori know T_PS. Thus (2.4) would be met by bounding the ambient space norm of M by the bound in (2.5). Also note that if M is constrained to lie in a region such that $|\mathcal{K}_{f,M}| \ \leq \ \alpha \ < |\mathcal{K}_{f,max}|$, then $\| \ \bar{x}_M \ \| \leq \sqrt{\frac{\gamma}{\alpha}}$ ensures that (2.3) is satisfied. And then following the same procedure as earlier, we would have

$$\varepsilon_M \ \leq \ \sqrt{\left(\frac{\gamma}{\alpha} + \frac{\gamma^2|\mathcal{K}_{f,low}|^2}{4\alpha^2}\right)} \Rightarrow \| \ \bar{x}_M \ \| \leq \sqrt{\frac{\gamma}{\alpha}}$$

2.2 Performance analysis of local reconstruction error: K points case

We now consider the case where $\mathcal{N}_\varepsilon(P) = \{P_1, ..., P_K\}$ $(K \ > \ 1)$.
Let P_i be $[x_1^{(i)} \ x_2^{(i)} \ f(x_1^{(i)}, x_2^{(i)})]$. We denote the orthogonal projection of P_i on T_PS by $\bar{x}_{P_i} \ = \ [x_1^{(i)} \ x_2^{(i)}]$.

$$\begin{aligned}\text{Thus, } E_{P_i}^2 \ &= \ \frac{1}{4}(<\bar{x}_{P_i}, \bar{v}_1>^2 \mathcal{K}_{f,max} + <\bar{x}_{P_i}, \bar{v}_2>^2 \mathcal{K}_{f,min})^2 \\ &= \ \frac{1}{4} \| \ \bar{x}_{P_i} \ \|^4 \ |\mathcal{K}_{f,P_i}|^2.\end{aligned}$$

Now if

$$\frac{1}{K}\sum_{i=1}^{K} E_{P_i}^2 \leq \frac{\gamma^2}{4} \tag{2.6}$$

$$\Leftrightarrow \sum_{i=1}^{K} \parallel \bar{x}_{P_i} \parallel^4 (r_{1,P_i}\mathcal{K}_{f,max} + r_{2,P_i}\mathcal{K}_{f,min})^2 \leq K\gamma^2,$$

$$\text{where} \qquad r_{1,P_i} = \frac{< \bar{x}_{P_i}, \bar{v_1} >^2}{\parallel \bar{x}_{P_i} \parallel^2}, \qquad r_{2,P_i} = \frac{< \bar{x}_{P_i}, \bar{v_2} >^2}{\parallel \bar{x}_{P_i} \parallel^2}.$$

We now consider the following disjoint subsets of $\mathcal{N}_\varepsilon(P)$ for $\alpha \in [|\mathcal{K}_{f,low}|, |\mathcal{K}_{f,max}|]$.

$$S_1 = \{P_i \in \mathcal{N}_\varepsilon(P) : |\mathcal{K}_{f,P_i}| \in (\alpha, |\mathcal{K}_{f,max}|]\},$$
$$S_2 = \{P_i \in \mathcal{N}_\varepsilon(P) : |\mathcal{K}_{f,P_i}| \in [|\mathcal{K}_{f,low}|, \alpha]\},$$

where $\mathcal{K}_{f,P_i} = r_{1,P_i}\mathcal{K}_{f,max} + r_{2,P_i}\mathcal{K}_{f,min}$ and $S_1 \cup S_2 = \mathcal{N}_\varepsilon(P)$. Observe that S_1 and S_2 correspond to points lying in the "high" and "low" curvature region respectively in $\mathcal{N}_\varepsilon(P)$.
Say, $|S_1| = K\delta$ and $|S_2| = K(1-\delta)$, for $0 \leq \delta \leq 1$.
Now

$$\sum_{i=1}^{K} \parallel \bar{x}_{P_i} \parallel^4 (r_{1,P_i}\mathcal{K}_{f,max} + r_{2,P_i}\mathcal{K}_{f,min})^2$$
$$\leq \mathcal{K}_{f,max}^2 \sum_{P_i \in S_1} \parallel \bar{x}_{P_i} \parallel^4 + \alpha^2 \sum_{P_i \in S_2} \parallel \bar{x}_{P_i} \parallel^4$$

Thus if we find conditions on the elements of S_1 and S_2 such that

$$\mathcal{K}_{f,max}^2 \sum_{P_i \in S_1} \parallel \bar{x}_{P_i} \parallel^4 + \alpha^2 \sum_{P_i \in S_2} \parallel \bar{x}_{P_i} \parallel^4 \leq K\gamma^2,$$

then those conditions are sufficient to satisfy (2.6). One can equivalently find conditions

$$\mathcal{K}_{f,max}^2 \sum_{P_i \in S_1} \parallel \bar{x}_{P_i} \parallel^4 \leq (K\gamma^2)w_1 \tag{2.7}$$

$$\alpha^2 \sum_{P_i \in S_2} \parallel \bar{x}_{P_i} \parallel^4 \leq (K\gamma^2)w_2 \tag{2.8}$$

where $w_1, w_2 \geq 0$ and $w_1 + w_2 = 1$. These conditions are sufficient to satisfy (2.6). We derive conditions on the elements of S_1 and S_2, which ensure that (2.7) and (2.8) are respectively satisfied with $w_1 = \delta$ and $w_2 = 1-\delta$. Note that we do so for the sake of avoiding more variables, however different values could also have been chosen.

In Section 2.2.1 we derive bounds on the norm of points in S_1 such that (2.7) is satisfied. In Section 2.2.2 we derive conditions on the angle for points in S_1. The angles are measured w.r.t the principal curvature directions, $\bar{v}_1$ and $\bar{v}_2$. Sections 2.2.3 and 2.2.4 contain analogous results for points in S_2.

2.2.1 Conditions on norms for points in S_1

Let $\| \bar{x}_{max,S_1} \|$ denote the maximum norm in the tangent space for a point in S_1.

$$\begin{aligned} \text{Say} \quad \mathcal{K}^2_{f,max} \sum_{P_i \in S_1} \| \bar{x}_{P_i} \|^4 &\leq (K\gamma^2)\delta \\ \text{But} \quad \mathcal{K}^2_{f,max} \sum_{P_i \in S_1} \| \bar{x}_{P_i} \|^4 &\leq \mathcal{K}^2_{f,max} \| \bar{x}_{max,S_1} \|^4 K\delta \\ \text{Therefore if} \quad \mathcal{K}^2_{f,max} \| \bar{x}_{max,S_1} \|^4 K\delta &\leq (K\gamma^2)\delta \\ \text{or equivalently} \quad \| \bar{x}_{max,S_1} \| &\leq \sqrt{\frac{\gamma}{|\mathcal{K}_{f,max}|}} \quad \text{holds,} \end{aligned} \tag{2.9}$$

then consequently (2.7) would be guaranteed, for $w_1 = \delta$. We now proceed to find the bound on the norm of the points in the ambient space, which ensures that (2.9) is satisfied. Let ε_{P_i,S_1} denote the ambient space norm of $P_i \in S_1$.

$$\begin{aligned} \text{Say } \varepsilon^2_{P_i,S_1} &\leq C_1^2 \\ \Rightarrow \frac{1}{4} \| \bar{x}_{P_i} \|^4 |\mathcal{K}_{f,P_i}|^2 + \| \bar{x}_{P_i} \|^2 &\leq C_1^2 \\ \Rightarrow \| \bar{x}_{P_i} \|^4 |\mathcal{K}_{f,P_i}|^2 + 4 \| \bar{x}_{P_i} \|^2 - 4C_1^2 &\leq 0 \\ \Rightarrow \| \bar{x}_{P_i} \|^2 &\leq \frac{2C_1^2}{1 + \sqrt{1 + |\mathcal{K}_{f,P_i}|^2 C_1^2}} \end{aligned}$$

Observe that since $\| \bar{x}_{P_i} \| \leq \sqrt{\frac{\gamma}{|\mathcal{K}_{f,max}|}}$, $\forall\ P_i \in S_1$, thus we need to find a condition on C_1 such that the following holds

$$\varepsilon^2_{P_i,S_1} \leq C_1^2 \Rightarrow \| \bar{x}_{P_i} \| \leq \sqrt{\frac{\gamma}{|\mathcal{K}_{f,max}|}}.$$

Now

$$
\begin{aligned}
\| \bar{x}_{P_i} \|^2 &\leq \frac{2C_1^2}{1+\sqrt{1+|\mathcal{K}_{f,P_i}|^2 C_1^2}} \\
&\leq \frac{2C_1^2}{1+\sqrt{1+\alpha^2 C_1^2}}
\end{aligned}
$$

$$
\begin{aligned}
\text{So if } \frac{2C_1^2}{1+\sqrt{1+\alpha^2 C_1^2}} &\leq \frac{\gamma}{|\mathcal{K}_{f,max}|} \\
\Rightarrow 2|\mathcal{K}_{f,max}|C_1^2 &\leq \gamma + \gamma\sqrt{1+\alpha^2 C_1^2} \\
\Rightarrow (2|\mathcal{K}_{f,max}|C_1^2 - \gamma)^2 &\leq \gamma^2(1+\alpha^2 C_1^2) \\
\Rightarrow C_1^2 &\leq \left(\frac{\gamma}{|\mathcal{K}_{f,max}|} + \frac{\gamma^2\alpha^2}{4|\mathcal{K}_{f,max}|^2} \right)
\end{aligned}
$$

$$
\text{Thus, } \varepsilon_{P_i,S_1} \;\leq\; \sqrt{\left(\frac{\gamma}{|\mathcal{K}_{f,max}|} + \frac{\gamma^2\alpha^2}{4|\mathcal{K}_{f,max}|^2} \right)} \tag{2.10}
$$

Hence if (2.10) is satisfied then (2.9) is guaranteed.

2.2.2 Conditions on angles for points in S_1

As per definition of S_1,

$$
\alpha < |\mathcal{K}_{f,P}| \leq |\mathcal{K}_{f,max}|, \;\forall\; P \;\in\; S_1. \tag{2.11}
$$

In this section we find conditions on (r_1, r_2) such that (2.11) is satisfied. We know that $|\mathcal{K}_{f,max}| > 0$ and $|\mathcal{K}_{f,max}| \;\geq\; |\mathcal{K}_{f,min}|$. Note that

$$
\alpha \;\in [|\mathcal{K}_{f,low}|, |\mathcal{K}_{f,max}|].
$$

Rewriting (2.11) and using $\mathcal{K}_{f,P} \;=\; r_1\mathcal{K}_{f,max} \;+\; r_2\mathcal{K}_{f,min}$, we get

$$
\begin{aligned}
r_1\mathcal{K}_{f,max} \;+\; r_2\mathcal{K}_{f,min} \;&\in\; [-|\mathcal{K}_{f,max}|, -\alpha) \cup (\alpha, |\mathcal{K}_{f,max}|] \\
\text{or} \quad r_1(\mathcal{K}_{f,max} - \mathcal{K}_{f,min}) + \mathcal{K}_{f,min} \;&\in\; [-|\mathcal{K}_{f,max}|, -\alpha) \cup (\alpha, |\mathcal{K}_{f,max}|].
\end{aligned}
$$

N.B: If $\mathcal{K}_{f,max} \;=\; \mathcal{K}_{f,min} \;=\; K_f$, then $\alpha \;=\; K_f$ and any pair $(r_1, r_2) \in [0,1]^2$ such that $r_1 + r_2 = 1$, would satisfy (2.11). So we consider the case $\mathcal{K}_{f,max} \neq \mathcal{K}_{f,min}$, for further analysis.

(1)$\mathcal{K}_{f,max}\mathcal{K}_{f,min} \geq 0$ *(i.e. same sign case)*

Note that $|\mathcal{K}_{f,low}| = |\mathcal{K}_{f,min}|$ in this case. We have the following cases.

(a)$\mathcal{K}_{f,max} > 0,\ \mathcal{K}_{f,min} \geq 0$

We have the following region for r_1

$$r_1 \in \left[\frac{-|\mathcal{K}_{f,max}| - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}, \frac{-\alpha - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}\right) \cup \left(\frac{\alpha - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}, \frac{|\mathcal{K}_{f,max}| - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}\right].$$

Taking the intersection with the region $0 \leq r_1 \leq 1$, we get

$$r_1 \in \left(\frac{\alpha - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}, 1\right].$$

(b)$\mathcal{K}_{f,max} < 0,\ \mathcal{K}_{f,min} \leq 0$

We have the following region for r_1

$$r_1 \in \left(\frac{\alpha + \mathcal{K}_{f,min}}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}, \frac{|\mathcal{K}_{f,max}| + \mathcal{K}_{f,min}}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}\right] \cup \left[\frac{\mathcal{K}_{f,min} - |\mathcal{K}_{f,max}|}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}, \frac{\mathcal{K}_{f,min} - \alpha}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}\right).$$

Taking the intersection with the region $0 \leq r_1 \leq 1$, we get

$$r_1 \in \left(\frac{\alpha + \mathcal{K}_{f,min}}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}, 1\right].$$

Thus from (a) and (b) we get the following conditions on r_1 and r_2 when $\mathcal{K}_{f,max}\mathcal{K}_{f,min} \geq 0$.

$$\begin{aligned} r_1 &\in \left(\frac{\alpha - |\mathcal{K}_{f,min}|}{|\mathcal{K}_{f,max}| - |\mathcal{K}_{f,min}|}, 1\right] \\ \text{and,}\quad r_2 &\in \left[0, \frac{|\mathcal{K}_{f,max}| - \alpha}{|\mathcal{K}_{f,max}| - |\mathcal{K}_{f,min}|}\right). \end{aligned}$$

(2)$\mathcal{K}_{f,max}\mathcal{K}_{f,min} \leq 0$ *(i.e. opposite sign case)*

Note that $|\mathcal{K}_{f,low}| = 0$ in this case. We have the following cases.

(c)$\mathcal{K}_{f,max} < 0,\ \mathcal{K}_{f,min} \geq 0$

We have the following region for r_1 (identical in structure to **1(b)**).

$$r_1 \in \left(\frac{\alpha + \mathcal{K}_{f,min}}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}, \frac{|\mathcal{K}_{f,max}| + \mathcal{K}_{f,min}}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}\right] \cup \left[\frac{\mathcal{K}_{f,min} - |\mathcal{K}_{f,max}|}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}, \frac{\mathcal{K}_{f,min} - \alpha}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}\right)$$

Taking the intersection with the region $0 \le r_1 \le 1$, we get the following.

$$\text{If } \alpha \in [0, |\mathcal{K}_{f,min}|) \Rightarrow r_1 \in \left(\frac{\alpha + \mathcal{K}_{f,min}}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}, 1\right] \cup \left[0, \frac{\mathcal{K}_{f,min} - \alpha}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}\right),$$

$$\text{and if } \alpha \in [|\mathcal{K}_{f,min}|, |\mathcal{K}_{f,max}|] \Rightarrow r_1 \in \left(\frac{\alpha + \mathcal{K}_{f,min}}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}, 1\right].$$

(d)$\mathcal{K}_{f,max} > 0,\ \mathcal{K}_{f,min} \le 0$

We have the following region for r_1 (identical in structure to **1(a)**).

$$r_1 \in \left[\frac{-|\mathcal{K}_{f,max}| - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}, \frac{-\alpha - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}\right) \cup \left(\frac{\alpha - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}, \frac{|\mathcal{K}_{f,max}| - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}\right].$$

Taking the intersection with the region $0 \le r_1 \le 1$, we get the following.

$$\text{If } \alpha \in [0, |\mathcal{K}_{f,min}|) \Rightarrow r_1 \in \left(\frac{\alpha - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}, 1\right] \cup \left[0, \frac{-\alpha - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}\right),$$

$$\text{and if } \alpha \in [|\mathcal{K}_{f,min}|, |\mathcal{K}_{f,max}|] \Rightarrow r_1 \in \left(\frac{\alpha - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}, 1\right]$$

Thus from (c) and (d) we get the following conditions on r_1 and r_2 when $\mathcal{K}_{f,max}\mathcal{K}_{f,min} \le 0$.

$$\begin{aligned}
&\text{If } \alpha \in [0, |\mathcal{K}_{f,min}|) \\
&\qquad \Rightarrow r_1 \in \left(\frac{\alpha + |\mathcal{K}_{f,min}|}{|\mathcal{K}_{f,max}| + |\mathcal{K}_{f,min}|}, 1\right] \cup \left[0, \frac{|\mathcal{K}_{f,min}| - \alpha}{|\mathcal{K}_{f,max}| + |\mathcal{K}_{f,min}|}\right), \\
&\qquad\quad r_2 \in \left(0, \frac{|\mathcal{K}_{f,max}| - \alpha}{|\mathcal{K}_{f,max}| + |\mathcal{K}_{f,min}|}\right] \cup \left(\frac{|\mathcal{K}_{f,max}| + \alpha}{|\mathcal{K}_{f,max}| + |\mathcal{K}_{f,min}|}, 1\right] \\
&\text{and if } \alpha \in [|\mathcal{K}_{f,min}|, |\mathcal{K}_{f,max}|] \\
&\qquad \Rightarrow r_1 \in \left(\frac{\alpha + |\mathcal{K}_{f,min}|}{|\mathcal{K}_{f,max}| + |\mathcal{K}_{f,min}|}, 1\right], \\
&\qquad\quad r_2 \in \left(0, \frac{|\mathcal{K}_{f,max}| - \alpha}{|\mathcal{K}_{f,max}| + |\mathcal{K}_{f,min}|}\right].
\end{aligned}$$

2.2.3 Conditions on norms for points in S_2

Let $\| \bar{x}_{max,S_2} \|$ denote the maximum norm in the tangent space for a point in S_2.

$$\begin{aligned} \text{Say} \quad & \alpha^2 \sum_{P_i \in S_2} \| \bar{x}_{P_i} \|^4 \leq (K\gamma^2)(1-\delta). \\ \text{But} \quad & \alpha^2 \sum_{P_i \in S_2} \| \bar{x}_{P_i} \|^4 \leq \alpha^2 \| \bar{x}_{max,S_2} \|^4 K(1-\delta). \\ \text{Therefore if} \quad & \alpha^2 \| \bar{x}_{max,S_2} \|^4 K(1-\delta) \leq (K\gamma^2)(1-\delta) \\ \text{or equivalently} \quad & \| \bar{x}_{max,S_2} \| \leq \sqrt{\frac{\gamma}{\alpha}} \quad \text{holds} \end{aligned} \tag{2.12}$$

then consequently (2.8) would be guaranteed for $w_2 = 1-\delta$. We now find the bound on the norms of the points in the ambient space, which ensures that (2.12) is satisfied. Let ε_{P_i,S_2} denote the ambient space norm of $P_i \in S_2$. Proceeding similarly as in Section 2.2.1, it is easy to show that

$$\varepsilon_{P_i,S_2} \leq \sqrt{\left(\frac{\gamma}{\alpha} + \frac{\gamma^2 \mathcal{K}_{f,low}^2}{4\alpha^2}\right)}. \tag{2.13}$$

Hence if (2.13) is satisfied then (2.12) is guaranteed.

2.2.4 Conditions on angles for points in S_2

As per definition of S_2,

$$|\mathcal{K}_{f,low}| \leq |\mathcal{K}_{f,P}| \leq \alpha, \ \forall \ P \in S_2 \tag{2.14}$$

In this section we find conditions on (r_1, r_2) such that (2.14) is satisfied. Rewriting (2.14) and using $\mathcal{K}_{f,P} = r_1\mathcal{K}_{f,max} + r_2\mathcal{K}_{f,min}$, we get the following.

$$\begin{aligned} r_1\mathcal{K}_{f,max} + r_2\mathcal{K}_{f,min} &\in [-\alpha, -|\mathcal{K}_{f,low}|] \cup [|\mathcal{K}_{f,low}|, \alpha] \\ \text{or,} \quad r_1(\mathcal{K}_{f,max} - \mathcal{K}_{f,min}) + \mathcal{K}_{f,min} &\in [-\alpha, -|\mathcal{K}_{f,low}|] \cup [|\mathcal{K}_{f,low}|, \alpha]. \end{aligned}$$

N.B: For the same reasons stated earlier, we consider the case $\mathcal{K}_{f,max} \neq \mathcal{K}_{f,min}$ for further analysis.

(1) $\mathcal{K}_{f,max}\mathcal{K}_{f,min} \geq 0$ *(i.e. same sign case)*

Note that $|\mathcal{K}_{f,low}| = |\mathcal{K}_{f,min}|$ in this case. We have the following cases.

(a) $\mathcal{K}_{f,max} > 0,\ \mathcal{K}_{f,min} \geq 0$

We have the following region for r_1.

$$r_1 \in \left[\frac{-\alpha - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}, \frac{-|\mathcal{K}_{f,low}| - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}\right] \cup \left[\frac{|\mathcal{K}_{f,low}| - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}, \frac{\alpha - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}\right].$$

Taking the intersection with the region $0 \ \leq r_1 \ \leq 1$, we get

$$r_1 \in \left[0, \frac{\alpha - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}\right]$$

(b) $\mathcal{K}_{f,max} < 0,\ \mathcal{K}_{f,min} \leq 0$

We have the following region for r_1.

$$r_1 \in \left[\frac{|\mathcal{K}_{f,low}| + \mathcal{K}_{f,min}}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}, \frac{\alpha + \mathcal{K}_{f,min}}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}\right] \cup \left[\frac{\mathcal{K}_{f,min} - \alpha}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}, \frac{\mathcal{K}_{f,min} - |\mathcal{K}_{f,low}|}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}\right].$$

Taking the intersection with the region $0 \ \leq r_1 \ \leq 1$, we get

$$r_1 \in \left[0, \frac{\alpha + \mathcal{K}_{f,min}}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}\right].$$

Thus from (a) and (b) we get the following conditions on r_1 and r_2 when $\mathcal{K}_{f,max}\mathcal{K}_{f,min} \geq 0$.

$$\begin{aligned} r_1 &\in \left[0, \frac{\alpha - |\mathcal{K}_{f,min}|}{|\mathcal{K}_{f,max}| - |\mathcal{K}_{f,min}|}\right] \\ \text{and,} \quad r_2 \ &\in \left[\frac{|\mathcal{K}_{f,max}| - \alpha}{|\mathcal{K}_{f,max}| - |\mathcal{K}_{f,min}|}, 1\right]. \end{aligned}$$

(2) $\mathcal{K}_{f,max}\mathcal{K}_{f,min} \leq 0$ *(i.e. opposite sign case)*

Note that $|\mathcal{K}_{f,low}| \ = \ 0$ in this case. We have the following cases.

(c) $\mathcal{K}_{f,max} < 0,\ \mathcal{K}_{f,min} \geq 0$

We have the following region for r_1 (identical in structure to **1(b)**).

$$r_1 \in \left[\frac{|\mathcal{K}_{f,low}| + \mathcal{K}_{f,min}}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}, \frac{\alpha + \mathcal{K}_{f,min}}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}\right] \cup \left[\frac{\mathcal{K}_{f,min} - \alpha}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}, \frac{\mathcal{K}_{f,min} - |\mathcal{K}_{f,low}|}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}\right].$$

Taking the intersection with the region $0 \ \le r_1 \ \le 1$, we get the following.

$$\text{If } \alpha \in [0, |\mathcal{K}_{f,min}|) \Rightarrow r_1 \in \left(\frac{\mathcal{K}_{f,min} - \alpha}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}}, \frac{\mathcal{K}_{f,min} + \alpha}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}} \right],$$
$$\text{and if } \alpha \in [|\mathcal{K}_{f,min}|, |\mathcal{K}_{f,max}|] \Rightarrow r_1 \in \left(0, \frac{\alpha + \mathcal{K}_{f,min}}{\mathcal{K}_{f,min} - \mathcal{K}_{f,max}} \right]$$

(d) $\mathcal{K}_{f,max} > 0$, $\mathcal{K}_{f,min} \le 0$

We have the following region for r_1 (identical in structure to **1(a)**).

$$r_1 \in \left[\frac{-\alpha - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}, \frac{-|\mathcal{K}_{f,low}| - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}} \right] \cup \left[\frac{|\mathcal{K}_{f,low}| - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}, \frac{\alpha - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}} \right].$$

Taking the intersection with the region $0 \ \le r_1 \ \le 1$, we get the following.

$$\text{If } \alpha \in [0, |\mathcal{K}_{f,min}|) \Rightarrow r_1 \in \left(\frac{-\alpha - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}}, \frac{\alpha - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}} \right],$$
$$\text{and if } \alpha \in [|\mathcal{K}_{f,min}|, |\mathcal{K}_{f,max}|] \Rightarrow r_1 \in \left[0, \frac{\alpha - \mathcal{K}_{f,min}}{\mathcal{K}_{f,max} - \mathcal{K}_{f,min}} \right].$$

Thus from (c) and (d) we get the following conditions on r_1 and r_2 when $\mathcal{K}_{f,max}\mathcal{K}_{f,min} \le 0$.

$$\begin{aligned}
&\text{If } \alpha \in [0, |\mathcal{K}_{f,min}|) \\
&\qquad r_1 \in \left(\frac{|\mathcal{K}_{f,min}| - \alpha}{|\mathcal{K}_{f,min}| + |\mathcal{K}_{f,max}|}, \frac{|\mathcal{K}_{f,min}| + \alpha}{|\mathcal{K}_{f,min}| + |\mathcal{K}_{f,max}|} \right] \\
&\qquad r_2 \in \left[\frac{|\mathcal{K}_{f,max}| - \alpha}{|\mathcal{K}_{f,min}| + |\mathcal{K}_{f,max}|}, \frac{|\mathcal{K}_{f,max}| + \alpha}{|\mathcal{K}_{f,min}| + |\mathcal{K}_{f,max}|} \right) \\
&\text{and if } \alpha \in [|\mathcal{K}_{f,min}|, |\mathcal{K}_{f,max}|] \\
&\qquad r_1 \in \left[0, \frac{\alpha + |\mathcal{K}_{f,min}|}{|\mathcal{K}_{f,min}| + |\mathcal{K}_{f,max}|} \right] \\
&\qquad r_2 \in \left[\frac{|\mathcal{K}_{f,max}| - \alpha}{|\mathcal{K}_{f,min}| + |\mathcal{K}_{f,max}|}, 1 \right]
\end{aligned}$$

Note: For simplicity of further analysis, we assume that $\alpha \in [|\mathcal{K}_{f,min}|, |\mathcal{K}_{f,max}|]$.

Conclusion: Thus to summarise we have the following sampling regions in $T_P S$.

$$M_{S_1} \in \left[0, \sqrt{\frac{\gamma}{|\mathcal{K}_{f,max}|}}\right],$$
$$M_{S_2} \in \left[0, \sqrt{\frac{\gamma}{\alpha}}\right],$$

where M_{S_1} and M_{S_2} denote the norms in the tangent space for points lying in S_1 and S_2 respectively.

$$\theta_{S_1,\bar{v}_1} \in (-\omega, \omega) \cup (\pi - \omega, \pi + \omega),$$
$$\theta_{S_2,\bar{v}_1} \in \left[\frac{\pi}{2} - \phi, \frac{\pi}{2} + \phi\right] \cup \left[-\frac{\pi}{2} - \phi, -\frac{\pi}{2} + \phi\right]$$

where $\theta_{S_1,\bar{v}_1}$ (resp. $\theta_{S_2,\bar{v}_1}$) denotes the angle made by an element of S_1 (resp. S_2) with $\bar{v}_1$. Also,

$$\omega = \begin{cases} \cos^{-1}\sqrt{\dfrac{\alpha - |\mathcal{K}_{f,min}|}{|\mathcal{K}_{f,max}| - |\mathcal{K}_{f,min}|}} & ; \ \mathcal{K}_{f,max}\mathcal{K}_{f,min} \geq 0 \\ \cos^{-1}\sqrt{\dfrac{\alpha + |\mathcal{K}_{f,min}|}{|\mathcal{K}_{f,max}| + |\mathcal{K}_{f,min}|}} & ; \ \mathcal{K}_{f,max}\mathcal{K}_{f,min} \leq 0 \end{cases}$$

$$\phi = \begin{cases} \cos^{-1}\sqrt{\dfrac{|\mathcal{K}_{f,max}| - \alpha}{|\mathcal{K}_{f,max}| - |\mathcal{K}_{f,min}|}} & ; \ \mathcal{K}_{f,max}\mathcal{K}_{f,min} \geq 0 \\ \cos^{-1}\sqrt{\dfrac{|\mathcal{K}_{f,max}| - \alpha}{|\mathcal{K}_{f,max}| + |\mathcal{K}_{f,min}|}} & ; \ \mathcal{K}_{f,max}\mathcal{K}_{f,min} \leq 0 \end{cases}$$

2.3 Random Sampling: local reconstruction error analysis

In the previous section we found *sufficient* conditions on the tangent space norms for the elements of S_1 and S_2. It is important to know the extent to which the radii of the regions for S_1 and S_2, (corresponding to the sufficient conditions) can be increased such that the average reconstruction error after sampling does not exceed $\frac{\gamma^2}{4}$ by more than a certain value. In other words if $\| \bar{x}_{\max,S_1} \| \leq \sqrt{\frac{a\gamma}{|\mathcal{K}_{f,max}|}}$ and $\| \bar{x}_{\max,S_2} \| \leq \sqrt{\frac{b\gamma}{\alpha}}$ then it is important to know by how much both a and b can be increased without increasing the average reconstruction error of the points in $\mathcal{N}_\varepsilon(P)$ drastically. In this section we derive conditions on a and b, and show that if those conditions are met, and also if the number of samples (K) in $\mathcal{N}_\varepsilon(P)$ is greater than a given lower bound, then it ensures that the probability of the "bad event" $\{\frac{1}{K}\sum_{i=1}^{K} E_{P_i}^2 \geq \frac{\gamma^2}{4}\}$ is upper bounded. Say we draw $K\delta$ points uniformly at random from S_1.

$$M_{S_1,i} \sim \mathrm{U}\left[0, \sqrt{\frac{a\gamma}{|\mathcal{K}_{f,max}|}}\right] \quad \text{i.i.d} \quad \forall P_i \in S_1$$
$$\theta_{S_1,i} \sim \mathrm{U}\{(-\omega,\omega) \cup (\pi-\omega, \pi+\omega)\} \quad \text{i.i.d} \quad \forall P_i \in S_1$$

where $M_{S_1,i}$ and $\theta_{S_1,i}$ are chosen independently $\forall P_i \in S_1$. U denotes uniform distribution. Say we draw $K(1-\delta)$ points uniformly at random from S_2.

$$M_{S_2,i} \sim \mathrm{U}\left[0, \sqrt{\frac{b\gamma}{\alpha}}\right] \quad \text{i.i.d} \quad \forall P_i \in S_2$$
$$\theta_{S_2,i} \sim \mathrm{U}\left\{\left[\frac{\pi}{2}-\phi, \frac{\pi}{2}+\phi\right] \cup \left[-\frac{\pi}{2}-\phi, -\frac{\pi}{2}+\phi\right]\right\} \quad \text{i.i.d} \quad \forall P_i \in S_2$$

where $M_{S_2,i}$ and $\theta_{S_2,i}$ are chosen independently $\forall P_i \in S_2$.U denotes uniform distribution. Assume that the points in S_1 are chosen independently from those in S_2. $a,\ b \in \mathbb{R}^+$ and $a, b < \infty$.

$$\text{Say,} \quad \bar{Y}_K = \frac{1}{K}\sum_{i=1}^{K} Y_i, \quad \text{where} \quad Y_i = E_{P_i}^2 = \frac{1}{4}M_i^4 \mathcal{K}_{f,P_i}^2$$

$$\text{Note that} \quad \mathbb{E}_{S_1}[Y_i] = \mathbb{E}_{S_1}\left[\frac{1}{4}M_i^4\mathcal{K}_{f,P_i}^2\right] = \frac{a^2\gamma^2}{5|\mathcal{K}_{f,max}|^2}E_1 = \mu_{S_1}, \tag{2.15}$$
$$\text{and} \quad \mathbb{E}_{S_2}[Y_i] = \mathbb{E}_{S_2}\left[\frac{1}{4}M_i^4\mathcal{K}_{f,P_i}^2\right] = \frac{b^2\gamma^2}{5\alpha^2}E_2 = \mu_{S_2}, \tag{2.16}$$
$$\text{where} \quad E_1 = \mathbb{E}_{\theta_{S_1}}\left[\frac{1}{4}\mathcal{K}_{f,P_i}^2\right] \Rightarrow \frac{1}{4}\alpha^2 \leq E_1 \leq \frac{1}{4}\mathcal{K}_{f,max}^2,$$
$$\text{and} \quad E_2 = \mathbb{E}_{\theta_{S_2}}\left[\frac{1}{4}\mathcal{K}_{f,P_i}^2\right] \Rightarrow \frac{1}{4}\mathcal{K}_{f,low}^2 \leq E_2 \leq \frac{1}{4}\alpha^2.$$

$\mathbb{E}_{S_1}[\cdot]$ and $\mathbb{E}_{S_2}[\cdot]$ denote expectations w.r.t the distributions of elements in S_1 and S_2 respectively. We have the following exact expressions for E_1 and E_2.

$$E_1 = \mathbb{E}_{\theta_{S_1}}\left[\frac{1}{4}\mathcal{K}^2_{f,P_i}\right]$$
$$= \frac{1}{4}(\mathcal{K}^2_{f,min} + A_2(\mathcal{K}_{f,max} - \mathcal{K}_{f,min})^2 + 2A_1(\mathcal{K}_{f,max} - \mathcal{K}_{f,min})\mathcal{K}_{f,min})$$

$$\text{where} \quad A_1 = \frac{1}{2}(1 + \text{sinc}(2\omega)),$$
$$\text{and} \quad A_2 = \frac{1}{8}(3 + 4\text{sinc}(2\omega) + \text{sinc}(4\omega)).$$

$$E_2 = \mathbb{E}_{\theta_{S_2}}\left[\frac{1}{4}\mathcal{K}^2_{f,P_i}\right]$$
$$= \frac{1}{4}(\mathcal{K}^2_{f,min} + B_2(\mathcal{K}_{f,max} - \mathcal{K}_{f,min})^2 + 2B_1(\mathcal{K}_{f,max} - \mathcal{K}_{f,min})\mathcal{K}_{f,min})$$

$$\text{where} \quad B_1 = \frac{1}{2}(1 - \text{sinc}(2\phi)),$$
$$\text{and} \quad B_2 = \frac{1}{8}(3 - 4\text{sinc}(2\phi) + \text{sinc}(4\phi)).$$

$$\text{Note that} \quad \bar{Y}_K = \delta\left(\frac{1}{K\delta}\sum_{P_i\in S_1} Y_i\right) + (1-\delta)\left(\frac{1}{K(1-\delta)}\sum_{P_i\in S_2} Y_i\right)$$

By the Strong Law of Large Numbers (SLLN),

$$\frac{1}{K\delta}\sum_{P_i\in S_1} Y_i \overset{K\to\infty}{\Longrightarrow} \mu_{S_1} \quad \text{a.s}$$
$$\frac{1}{K(1-\delta)}\sum_{P_i\in S_2} Y_i \overset{K\to\infty}{\Longrightarrow} \mu_{S_2} \quad \text{a.s}$$
$$\text{Thus,} \quad \bar{Y}_K \overset{K\to\infty}{\Longrightarrow} \delta\mu_{S_1} + (1-\delta)\mu_{S_2} = \mu \quad \text{a.s}$$

From (2.15) and (2.16) we have

$$\mu \;=\; \frac{\gamma^2}{5}\left[\delta\left(\frac{a^2 E_1}{|\mathcal{K}_{f,max}|^2}\right) + (1-\delta)\left(\frac{b^2 E_2}{\alpha^2}\right)\right]$$

$$\text{Thus,} \quad \frac{\gamma^2}{20}\left[\delta\left(\frac{a^2\alpha^2}{|\mathcal{K}_{f,max}|^2}\right) + (1-\delta)\left(\frac{b^2\mathcal{K}^2_{f,low}}{\alpha^2}\right)\right] \le \mu \le \frac{\gamma^2}{20}\left[\delta a^2 + (1-\delta)b^2\right]$$

N.B: Note that as a and/or b increases, μ increases and eventually can be more than $\frac{\gamma^2}{4}$. Thus it is natural to define conditions on a and b such that

$$\mathbb{P}\left(\left\{\bar{Y}_K \geq \frac{\gamma^2}{4} + \epsilon\right\}\right) \ < \ \beta \ < 1 \quad \text{for some} \quad \epsilon > 0,$$

where the probability measure $\mathbb{P}$ is induced by the the joint distribution of elements of S_1 and S_2.

$$\begin{aligned}
\mathbb{P}\left(\left\{\bar{Y}_K \geq \frac{\gamma^2}{4} + \epsilon\right\}\right) &= \mathbb{P}\left(\left\{\sum_{i=1}^{K} Y_i \geq K\left(\frac{\gamma^2}{4} + \epsilon\right)\right\}\right) \\
&= \mathbb{P}\left(\left\{ e^{t\sum_{i=1}^{K} Y_i} \geq e^{tK(\frac{\gamma^2}{4} + \epsilon)} \right\}\right) \qquad (\forall\ t \geq 0) \\
&\leq e^{-tK\left(\frac{\gamma^2}{4} + \epsilon\right)} \mathbb{E}\left[e^{t\sum_{i=1}^{K} Y_i} \right] \qquad \text{(Markov's Inequality)} \\
&= e^{-tK\left(\frac{\gamma^2}{4} + \epsilon\right)} \left(\mathbb{E}_{S_1}\left[e^{tY_i}\right]\right)^{K\delta} \left(\mathbb{E}_{S_2}\left[e^{tY_i}\right]\right)^{K(1-\delta)}
\end{aligned}$$

where the last expression follows from the independency of P_i's in S_1 and S_2. We now consider the following cases for deriving conditions on a and b.

(i) *Trivial conditions for a and b*

$$\mathbb{E}_{S_1}\left[e^{tY_i}\right] \leq e^{t\frac{a^2\gamma^2}{4}} \quad \left(Y_i \leq \frac{a^2\gamma^2}{4}; \quad \forall \quad P_i \in S_1\right)$$

$$\mathbb{E}_{S_2}\left[e^{tY_i}\right] \leq e^{t\frac{b^2\gamma^2}{4}} \quad \left(Y_i \leq \frac{b^2\gamma^2}{4}; \quad \forall \quad P_i \in S_2\right)$$

$$\begin{aligned}
\therefore \mathbb{P}\left(\left\{\bar{Y}_K \geq \frac{\gamma^2}{4} + \epsilon\right\}\right) &\leq e^{-tK\left(\frac{\gamma^2}{4} + \epsilon\right)} . e^{tK\delta\frac{a^2\gamma^2}{4}} . e^{tK(1-\delta)\frac{b^2\gamma^2}{4}} \\
&= e^{-tK\left[\left(\frac{\gamma^2}{4} + \epsilon\right) - \frac{\gamma^2}{4}(\delta a^2 + (1-\delta)b^2)\right]} \qquad (2.17)
\end{aligned}$$

Observe that for a non-trivial bound in (2.17), we require

$$\frac{\gamma^2}{4}(\delta a^2 + (1-\delta)b^2) < \frac{\gamma^2}{4} + \epsilon,$$
$$\Leftrightarrow \delta a^2 + (1-\delta)b^2 < 1 + \frac{4\epsilon}{\gamma^2}. \tag{2.18}$$

Hence if a and b are chosen such that (2.18) is satisfied for some fixed $\epsilon > 0$ then (2.17) will be strictly less than 1 and exponentially decreasing w.r.t t. Assuming that a and b satisfy (2.18) and since (2.17) holds $\forall\ t \geq 0$, we have

$$\begin{aligned}\mathbb{P}\left(\left\{\bar{Y}_K \geq \frac{\gamma^2}{4} + \epsilon\right\}\right) &\leq \inf_{t\geq 0} e^{-tK\left[\left(\frac{\gamma^2}{4}+\epsilon\right) - \frac{\gamma^2}{4}(\delta a^2 + (1-\delta)b^2)\right]} \\ &= 0\end{aligned}$$

(ii) *Non trivial conditions for a and b*

$$\mathbb{E}_{S_1}[e^{tY_i}] = \mathbb{E}_{S_1}[e^{t(Y_i - \mu_{S_1})}]e^{t\mu_{S_1}}.$$

$$\begin{aligned}\text{But} \quad & -\mu_{S_1} \leq (Y_i - \mu_{S_1}) \leq \frac{1}{4}a^2\gamma^2 - \mu_{S_1} \quad (\forall P_i\ \in S_1), \\ \text{where} \quad & \mathbb{E}_{S_1}[Y_i] = \mu_{S_1} \quad (\forall P_i \in S_1).\end{aligned}$$

$$\begin{aligned}\text{Thus, we have} \quad \mathbb{E}_{S_1}[e^{t(Y_i - \mu_{S_1})}] &\leq e^{\frac{t^2a^4\gamma^4}{128}}, \quad \text{(Hoeffding's Lemma)} \\ \therefore \quad \mathbb{E}_{S_1}[e^{tY_i}] &\leq e^{\frac{t^2a^4\gamma^4}{128} + t\mu_{S_1}}. \\ \text{Similarly,} \quad \mathbb{E}_{S_2}[e^{tY_i}] &\leq e^{\frac{t^2b^4\gamma^4}{128} + t\mu_{S_2}}.\end{aligned}$$

So,

$$\begin{aligned}\mathbb{P}\left(\left\{\bar{Y}_K \geq \frac{\gamma^2}{4} + \epsilon\right\}\right) &\leq e^{-tK\left(\frac{\gamma^2}{4}+\epsilon\right)}.e^{K\delta\left(\frac{t^2a^4\gamma^4}{128} + t\mu_{S_1}\right)}.e^{K(1-\delta)\left(\frac{t^2b^4\gamma^4}{128} + t\mu_{S_2}\right)} \\ &= e^{\{\frac{t^2K\gamma^4}{128}(\delta a^4 + (1-\delta)b^4) + tK[(\delta\mu_{S_1} + (1-\delta)\mu_{S_2}) - \left(\frac{\gamma^2}{4}+\epsilon\right)]\}}\end{aligned}$$

Note that to have a non-trivial bound we require

$$\delta\mu_{S_1} + (1-\delta)\mu_{S_2} < \frac{\gamma^2}{4} + \epsilon,$$
$$\text{or} \quad \frac{\gamma^2}{5}\left[\delta\left(\frac{a^2 E_1}{|\mathcal{K}_{f,max}|^2}\right) + (1-\delta)\left(\frac{b^2 F_1}{\alpha^2}\right)\right] < \frac{\gamma^2}{4} + \epsilon. \tag{2.19}$$

Thus if a and b are chosen to satisfy (2.19) then it ensures that a non-trivial upper bound on $\mathbb{P}(\{\bar{Y}_K \geq \frac{\gamma^2}{4} + \epsilon\})$ exists. Furthermore, note that

$$\delta\mu_{S_1} + (1-\delta)\mu_{S_2} < \frac{\gamma^2}{20}\left[\delta a^2 + (1-\delta)b^2\right]$$

Thus (2.19) is also guaranteed if

$$\frac{\gamma^2}{20}\left[\delta a^2 + (1-\delta)b^2\right] < \frac{\gamma^2}{4} + \epsilon$$
$$\Leftrightarrow \delta a^2 + (1-\delta)b^2 < \left(5 + \frac{20\epsilon}{\gamma^2}\right). \tag{2.20}$$

We now assume that a and b are chosen to satisfy (2.19). Observe that,

$$\mathbb{P}\left(\left\{\bar{Y}_K \geq \frac{\gamma^2}{4} + \epsilon\right\}\right) \leq \inf_{t\geq 0} e^{\{\frac{t^2 K\gamma^4}{128}(\delta a^4 + (1-\delta)b^4) + tK[(\delta\mu_{S_1} + (1-\delta)\mu_{S_2}) - \left(\frac{\gamma^2}{4} + \epsilon\right)]\}} \tag{2.21}$$

The value of t which minimizes (2.21) is

$$t = \frac{64\left[\left(\frac{\gamma^2}{4} + \epsilon\right) - \mu\right]}{\gamma^4(\delta a^4 + (1-\delta)b^4)}$$

Putting this value in (2.21) we get

$$\mathbb{P}\left(\left\{\bar{Y}_K \geq \frac{\gamma^2}{4} + \epsilon\right\}\right) \leq e^{\left[-\frac{32K}{\gamma^4}\left(\left(\frac{\gamma^2}{4} + \epsilon\right) - \mu\right)^2 \frac{1}{(\delta a^4 + (1-\delta)b^4)}\right]}$$

Thus, $\mathbb{P}\left(\left\{\bar{Y}_K \geq \frac{\gamma^2}{4} + \epsilon\right\}\right) < \beta < 1$ is ensured if

$$e^{\left[-\frac{32K}{\gamma^4}\left(\left(\frac{\gamma^2}{4}+\epsilon\right)-\mathbb{E}[\bar{Y}_K]\right)^2 \frac{1}{(\delta a^4+(1-\delta)b^4)}\right]} < \beta$$

$$\Leftrightarrow \quad K > \frac{\gamma^4(\delta a^4 + (1-\delta)b^4)ln\left(\frac{1}{\beta}\right)}{32\left(\frac{\gamma^2}{4}+\epsilon-\mu\right)^2}$$

$$= K_{\text{bound}}.$$

Conclusion: If we sample K points uniformly and independently at random from the regions corresponding to S_1 and S_2 such that for some $0 < \delta < 1$ and $0 < \beta < 1$

(a) $|S_1| = K\delta$, $|S_2| = K(1-\delta)$,

(b) $\| \bar{x}_{\max,S_1} \| \leq \sqrt{\frac{a\gamma}{|\mathcal{K}_{f,max}|}}$, $\| \bar{x}_{\max,S_2} \| \leq \sqrt{\frac{b\gamma}{\alpha}}$ where a and b satisfy

$$\delta\left(\frac{a^2E_1}{|\mathcal{K}_{f,max}|^2}\right) + (1-\delta)\left(\frac{b^2E_2}{\alpha^2}\right) < \frac{5}{4} + \frac{5\epsilon}{\gamma^2},$$

where $E_1 = \mathbb{E}_{\theta_{S_1}}[\frac{1}{4}\mathcal{K}_{f,P}]$, $E_2 = \mathbb{E}_{\theta_{S_2}}[\frac{1}{4}\mathcal{K}_{f,P}]$

(c) $K > \dfrac{\gamma^4(\delta a^4 + (1-\delta)b^4)ln\left(\frac{1}{\beta}\right)}{32\left(\frac{\gamma^2}{4}+\epsilon-\mu\right)^2}$, where $\mu = \frac{\gamma^2}{5}\left[\delta\left(\frac{a^2E_1}{|\mathcal{K}_{f,max}|^2}\right) + (1-\delta)\left(\frac{b^2E_2}{\alpha^2}\right)\right]$

then the probability of the average reconstruction error of the points in $\mathcal{N}_\varepsilon(P)$ exceeding $\frac{\gamma^2}{4} + \epsilon$ will be at most β.

2.4 Experiment Results: local reconstruction error analysis

In this section we present results for the behaviour of the average reconstruction error w.r.t variation of the sampling parameters α, δ and (a, b). Two surfaces representing different possible neighbourhoods of a point were considered, one with principal curvature values having the same sign, and the other with opposite sign principal curvature values. The points were generated uniformly at random in the same manner as described in Section 2.3.

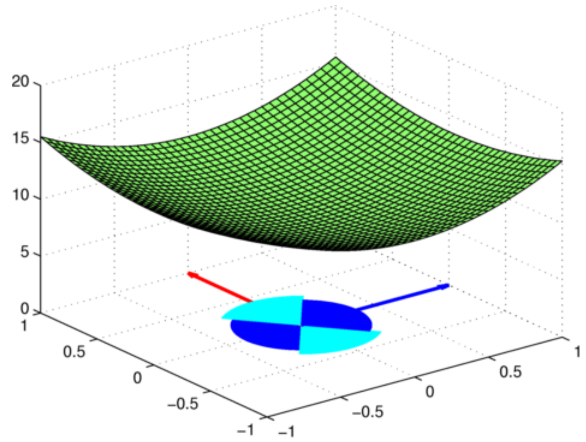

Figure 2.1: Surface with $\mathcal{K}_{f,max}$= 7, $\mathcal{K}_{f,min}$= 4. The blue (resp. red) axis denotes the direction of $\mathcal{K}_{f,max}$(resp. $\mathcal{K}_{f,min}$). S_1 and S_2 are shown respectively as blue and cyan shaded sectors of circles. $\alpha = 5.18$ was used for creating S_1 and S_2.

In Figs. 2.1 and 2.2, $(a, b) = (1, 1)$ and $(\gamma, \epsilon) = (1, 0)$ were used for generating the sampling regions. We now present simulation results as each of the sampling parameters is varied. For the parameter (a, b), we choose to keep $a = b$ throughout, for the sake of avoiding more variables. $\bar{E}_K$ is used to denote the empirical average reconstruction error. For computing $\bar{E}_K$, we choose K such that it is the smallest value greater than K_{bound} and is also a multiple of 10. $\beta = 0.1$ and $(\gamma, \epsilon) = (1, 0)$ were fixed in all the experiments. Furthermore $\bar{E}_K$ is averaged over 50 trials for each value of K.

2.4.1 Behaviour of K_{bound} and $\bar{E}_K$ as a is varied

In this experiment, we keep δ, α fixed, and observe the behaviour of K_{bound} and $\bar{E}_K$ as a is varied. We choose different values for a, each satisfying

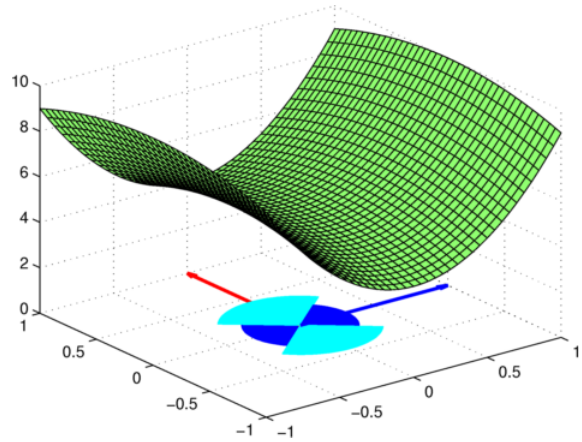

Figure 2.2: Surface with $\mathcal{K}_{f,max}$= 10, $\mathcal{K}_{f,min}$= -2. The blue (resp. red) axis denotes direction of $\mathcal{K}_{f,max}$(resp. $\mathcal{K}_{f,min}$). S_1 and S_2 are shown respectively as blue and cyan shaded sectors of circles. $\alpha = 5.06$ was used for creating S_1 and S_2.

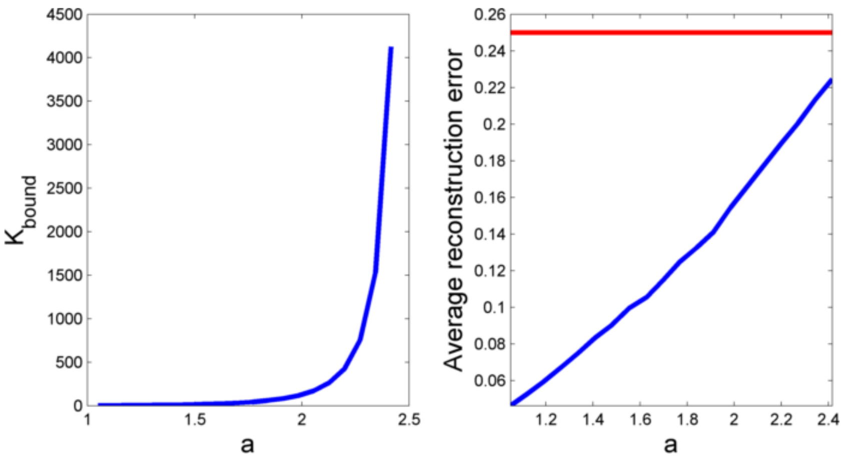

Figure 2.3: $\mathcal{K}_{f,max} = 7$, $\mathcal{K}_{f,min} = 4$, $\alpha = 5.42$, $\delta = 0.5$. The reconstruction error threshold $\frac{\gamma^2}{4} + \epsilon = 0.25$, is marked in red.

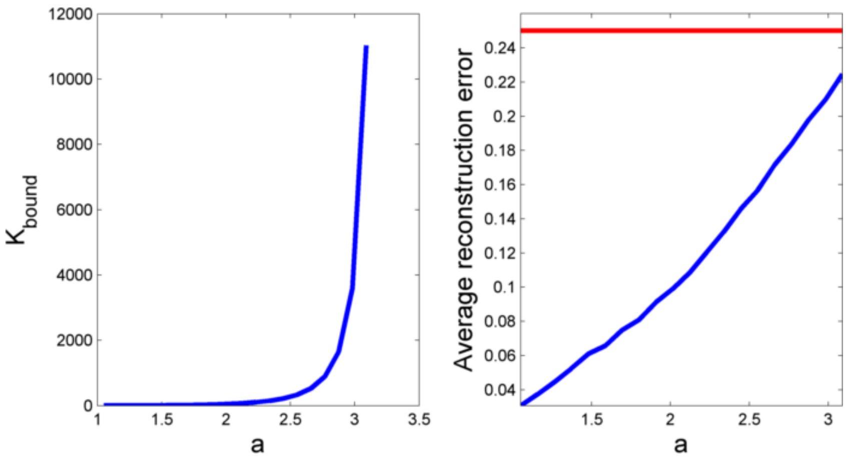

Figure 2.4: $\mathcal{K}_{f,max} = 10$, $\mathcal{K}_{f,min} = -2$, $\alpha = 5.8$, $\delta = 0.5$. The reconstruction error threshold $\frac{\gamma^2}{4} + \epsilon = 0.25$, is marked in red.

$$1 < a < \left(\frac{\frac{5}{4} + \frac{5\epsilon}{\gamma^2}}{\delta \frac{E_1}{|\mathcal{K}_{f,max}^2|} + (1-\delta)\frac{E_2}{\alpha^2}} \right)^{\frac{1}{2}}.$$

The above condition is obtained by putting $a = b$ in the conditions derived in Section 2.3. E_1 and E_2 are functions of α as defined earlier. We see from Figs. 2.3,2.4 that K_{bound} increases slowly till a $\simeq 2$, after which it increases at a faster rate. $\bar{E}_K$ increases steadily with a, which is expected since we are gradually sampling from larger regions.

2.4.2 Behaviour of K_{bound} and $\bar{E}_K$ as α is varied

In this experiment, we keep δ, a fixed, and observe the behaviour of K_{bound} and $\bar{E}_K$ as α is varied. Since in this experiment α is varying, we choose a s.t the following is satisfied:

$$1 < a < \left(5 + \frac{20\epsilon}{\gamma^2}\right)^{\frac{1}{2}}$$

Note that any value of a satisfying the above condition ensures: $\mathbb{P}(\{\bar{Y}_K \geq \frac{\gamma^2}{4} + \epsilon\}) < \beta$,as was shown in Section 2.3. We see from Figs. 2.5,2.6 that as α increases, K_{bound} first decreases and then starts increasing for both the surfaces. A similar pattern can be seen for $\bar{E}_K$.

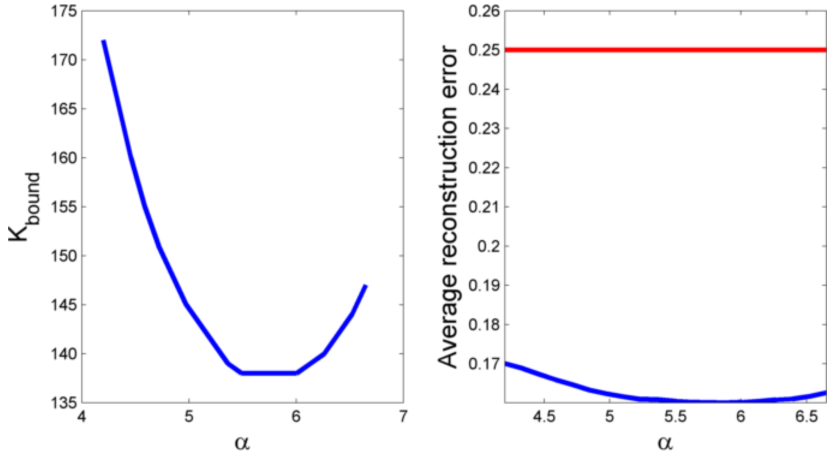

Figure 2.5: $\mathcal{K}_{f,max} = 7$, $\mathcal{K}_{f,min} = 4$, $a = 2.01$, $\delta = 0.5$. The reconstruction error threshold $\frac{\gamma^2}{4} + \epsilon = 0.25$, is marked in red.

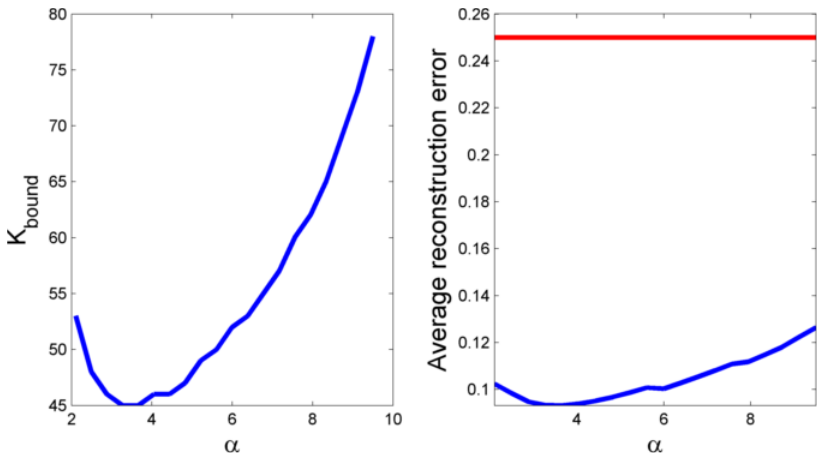

Figure 2.6: $\mathcal{K}_{f,max} = 10$, $\mathcal{K}_{f,min} = -2$, $a = 2.01$, $\delta = 0.5$. The reconstruction error threshold $\frac{\gamma^2}{4} + \epsilon = 0.25$, is marked in red.

2.4.3 Behaviour of K_{bound} and $\bar{E}_K$ as δ is varied

In this experiment, we keep α, a constant, and observe the behaviour of K_{bound} and $\bar{E}_K$ as δ is varied. Since δ is varying in this experiment, we choose

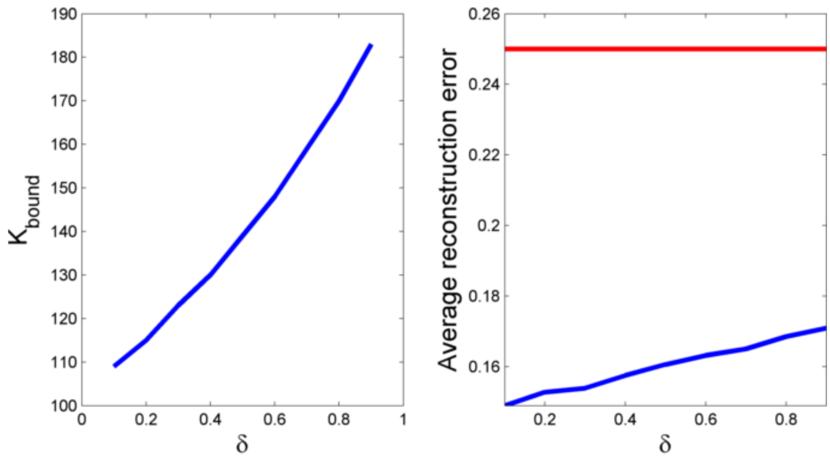

Figure 2.7: $\mathcal{K}_{f,max} = 7$, $\mathcal{K}_{f,min} = 4$, $a = 2.01$, $\alpha = 5.42$. The reconstruction error threshold $\frac{\gamma^2}{4} + \epsilon = 0.25$, is marked in red.

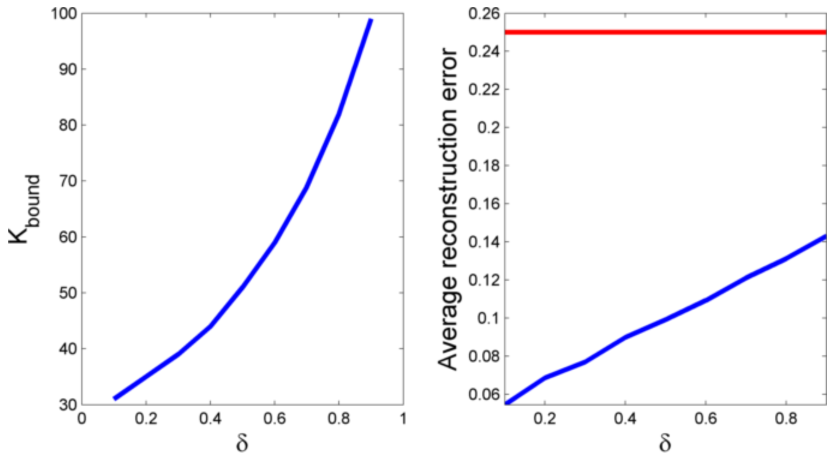

Figure 2.8: $\mathcal{K}_{f,max} = 10$, $\mathcal{K}_{f,min} = -2$, $a = 2.01$, $\alpha = 5.8$. The reconstruction error threshold $\frac{\gamma^2}{4} + \epsilon = 0.25$, is marked in red.

a the same way as in 2.4.2. We see from Figs. 2.7,2.8 that K_{bound} increases monotonically with δ. The value of α was chosen to lie approximately close to the average of $|\mathcal{K}_{f,min}|$ and $|\mathcal{K}_{f,max}|$. By increasing the fraction of total samples lying in the "high" curvature region, the total number of samples needed to guarantee the upper bound on $\mathbb{P}(\{\bar{Y}_K \geq \frac{\gamma^2}{4} + \epsilon\})$ should intuitively increase, and this is what we observe. The same reasoning applies to the behaviour observed for $\bar{E}_K$.

2.5 Performance analysis of optimal locally estimated linear subspace

In this section we derive conditions on the points in $\mathcal{N}_\varepsilon(P)$ such that the "angle" between the tangent space $T_P S$ and its estimation $\hat{T}_P S$ is upper bounded. $\hat{T}_P S$ is a 2-dimensional linear subspace optimal in the least squares sense. It is estimated using the points in $\mathcal{N}_\varepsilon(P)$. The notion of "angle" we use is as defined in [2]. Say the points are formed by sampling from within a disc of radius $\sqrt{\frac{\eta}{|\mathcal{K}_{f,max}|}}$ in $T_P S$. We show that a bound on the "angle" is guaranteed provided that η itself is upper bounded. Again with the same notation as in the previous section, assume we have K points in the neighbourhood of P.

$$\text{Let } X = \begin{bmatrix} x_1 & x_2 & \cdots & x_K \\ y_1 & y_2 & \cdots & y_K \\ f(x_1, y_1) & f(x_2, y_2) & \cdots & f(x_K, y_K) \end{bmatrix}$$

The optimal 2 dimensional linear subspace (in the quadratic error sense) passing through P will be the one spanned by the 2 largest eigenvectors i.e. the eigenvectors corresponding to the 2 largest eigenvalues of

$$XX^T = \begin{bmatrix} \sum_i x_i^2 & \sum_i x_i y_i & \sum_i x_i f(x_i, y_i) \\ \sum_i y_i x_i & \sum_i y_i^2 & \sum_i y_i f(x_i, y_i) \\ \sum_i x_i f(x_i, y_i) & \sum_i y_i f(x_i, y_i) & \sum_i f^2(x_i, y_i) \end{bmatrix} = U \Lambda U^T,$$

where

$$U = [\bar{u}_1 \ \bar{u}_2 \ \bar{u}_3], \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Assume that $\lambda_1 \geq \lambda_2 > 0$ i.e., the rank of XX^T is at least 2. Say $\lambda_2 > \lambda_3$. Note that $\lambda_3 \geq 0$.
In [2], the angle between two subspaces $A = \text{span}\{\bar{a}_1, \ldots, \bar{a}_p\}$ and $B = \text{span}\{\bar{b}_1, \ldots, \bar{b}_q\}$ of a Euclidean space $\mathbb{R}^n$ is defined as

$$\cos^2 \theta := \det(M^T M)$$

where $\bar{a}_i$'s and $\bar{b}_i$'s are orthonormal. $[M^T]_{i,k} := [< \bar{a}_i, \bar{b}_k >]$ is a $p \times q$ matrix (with $1 \leq p \leq q$) and $< \cdot, \cdot >$ denotes the usual inner product on $\mathbb{R}^n$.In our case

$$\hat{T}_P S = \mathrm{span}\{\bar{u}_1, \bar{u}_2\}, \qquad T_P S = \mathrm{span}\{\bar{e}_1, \bar{e}_2\},$$

where

$$\begin{aligned} \bar{u}_1 &= [u_{11}\ u_{12}\ u_{13}]^T, \qquad \bar{u}_2 = [u_{21}\ u_{22}\ u_{23}]^T, \\ \bar{e}_1 &= [1\ 0\ 0]^T, \qquad \bar{e}_2 = [0\ 1\ 0]^T, \end{aligned}$$

and $< \bar{u}_i, \bar{u}_j > = \delta_{ij};\ i, j = 1, 2$. We first derive the expression for θ.

Let $U^{(2)} = \begin{bmatrix} \bar{u}_1\ \bar{u}_2 \end{bmatrix}, \qquad E = \begin{bmatrix} \bar{e}_1\ \bar{e}_2 \end{bmatrix}.$

$$\begin{aligned} \text{As,} \qquad M^T &= U^{(2)^T} E \\ \Rightarrow \qquad M^T M &= U^{(2)^T} E E^T U, \\ \text{where} \qquad E E^T &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} \text{Thus,} \qquad M^T M &= \begin{bmatrix} \bar{u}_1^T \\ \bar{u}_2^T \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_1\ \bar{u}_2 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \end{bmatrix} \begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} u_{11}^2 + u_{12}^2 & u_{11}u_{21} + u_{12}u_{22} \\ u_{11}u_{21} + u_{12}u_{22} & u_{21}^2 + u_{22}^2 \end{bmatrix} = \begin{bmatrix} 1 - u_{13}^2 & -u_{13}u_{23} \\ -u_{13}u_{23} & 1 - u_{23}^2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{Hence,} \qquad \cos^2\theta := \det M^T M &= (1 - u_{13}^2)(1 - u_{23}^2) - u_{13}^2 u_{23}^2 \\ &= 1 - u_{13}^2 - u_{23}^2. \end{aligned}$$

Thus we see that in order to upper bound θ we equivalently need to upper bound $|u_{13}|^2 + |u_{23}|^2$. We now analyse the conditions under which this is possible. Furthermore, we derive an upper bound only for $|u_{23}|$ and show later that it is sufficient for our purpose.

Consider $XX^T \bar{u}_2 = \lambda_2 \bar{u}_2$.

$$\begin{aligned} (\sum_i x_i^2)u_{21} + (\sum_i x_i y_i)u_{22} + \frac{1}{2}(\sum_i x_i \parallel \bar{x}_i \parallel^2 \mathcal{K}_{f,P_i})u_{23} &= \lambda_2 u_{21} \\ (\sum_i y_i x_i)u_{21} + (\sum_i y_i^2)u_{22} + \frac{1}{2}(\sum_i y_i \parallel \bar{x}_i \parallel^2 \mathcal{K}_{f,P_i})u_{23} &= \lambda_2 u_{22} \\ \frac{1}{2}(\sum_i x_i \parallel \bar{x}_i \parallel^2 \mathcal{K}_{f,P_i})u_{21} + \frac{1}{2}(\sum_i y_i \parallel \bar{x}_i \parallel^2 \mathcal{K}_{f,P_i})u_{22} + \frac{1}{4}(\sum_i \parallel \bar{x}_i \parallel^4 \mathcal{K}_{f,P_i}^2)u_{23} &= \lambda_2 u_{23} \end{aligned} \tag{2.22}$$

Taking the modulus of both sides of (2.22) we have the following.

$$
\begin{aligned}
\lambda_2|u_{23}| \leq \frac{1}{2}(|\sum_i x_i \parallel \bar{x}_i \parallel^2 \mathcal{K}_{f,P_i}|)|u_{21}|+ \\
\frac{1}{2}(|\sum_i y_i \parallel \bar{x}_i \parallel^2 \mathcal{K}_{f,P_i}|)|u_{22}| + \frac{1}{4}(|\sum_i \parallel \bar{x}_i \parallel^4 \mathcal{K}^2_{f,P_i}|)|u_{23}| \\
\Rightarrow (\lambda_2 - \frac{1}{4}\sum_i \parallel \bar{x}_i \parallel^4 |\mathcal{K}_{f,P_i}|^2)|u_{23}| \leq \frac{1}{2}\sum_i |x_i| \parallel \bar{x}_i \parallel^2 |\mathcal{K}_{f,P_i}|+ \\
\frac{1}{2}\sum_i |y_i| \parallel \bar{x}_i \parallel^2 |\mathcal{K}_{f,P_i}| \\
= \frac{1}{2}\sum_i \parallel \bar{x}_i \parallel^2 |\mathcal{K}_{f,P_i}|(|x_i| + |y_i|)
\end{aligned}
$$

An upper bound for $|u_{23}|$ will exist if $\lambda_2 \ - \ \frac{1}{4}\sum_i \parallel \bar{x}_i \parallel^4 |\mathcal{K}_{f,P_i}|^2 \ > \ 0.$

Note that

$$
\begin{aligned}
Tr(XX^T) &= \lambda_1 + \lambda_2 + \lambda_3 \\
&= \sum_i(\parallel \bar{x}_i \parallel^2 + f^2(x_i, y_i)).
\end{aligned}
$$

We assume now that points are sampled such that $\frac{\lambda_1}{\lambda_2} \ < \ r$ where $r \geq 1$ is a fixed value.

As $\lambda_1 + \lambda_2 + \lambda_3 \ \leq (r+2)\lambda_2$, we have

$$
\begin{aligned}
\lambda_2 &\geq \frac{1}{r+2}(\sum_i(\parallel \bar{x}_i \parallel^2 + f^2(x_i, y_i))), \\
\Rightarrow \qquad \lambda_2 \ - \ \frac{1}{4}\sum_i \parallel \bar{x}_i \parallel^4 |\mathcal{K}_{f,P_i}|^2 &\geq \frac{1}{r+2}\sum_i(\parallel \bar{x}_i \parallel^2 (1 - \frac{(r+1)}{4}|\mathcal{K}_{f,P_i}|^2 \parallel \bar{x}_i \parallel^2)), \\
&\geq \frac{1}{r+2}\sum_i(\parallel \bar{x}_i \parallel^2 (1 - \frac{(r+1)}{4}|\mathcal{K}_{f,max}|^2 \parallel \bar{x}_{max,\mathcal{N}_\varepsilon(P)} \parallel^2)), \\
&\geq \frac{1}{r+2}\left(1 - \frac{(r+1)}{4}|\mathcal{K}_{f,max}|\eta\right)\sum_i(\parallel \bar{x}_i \parallel^2).
\end{aligned}
$$

where $\parallel \bar{x}_{max,\mathcal{N}_\varepsilon(P)} \parallel = \sqrt{\dfrac{\eta}{|\mathcal{K}_{f,max}|}}$. In order to have $\lambda_2 - \frac{1}{4}\sum_i \parallel \bar{x}_i \parallel^4 |\mathcal{K}_{f,P_i}|^2 > 0$ it is sufficient to have

$$\left(1 - \frac{(r+1)}{4}|\mathcal{K}_{f,max}|\eta\right) > 0,$$

$$\text{or equivalently} \quad \eta < \left(\frac{4}{(r+1)|\mathcal{K}_{f,max}|}\right). \tag{2.23}$$

We see clearly that (2.23) also guarantees $\lambda_1 - \frac{1}{4}\sum_i \parallel \bar{x}_i \parallel^4 |\mathcal{K}_{f,P_i}|^2 > 0$. So assuming that η satisfies (2.23) we now proceed to find an upper bound on $|u_{23}|$.

$$|u_{23}| \leq \frac{\dfrac{1}{2}\displaystyle\sum_i \parallel \bar{x}_i \parallel^2 |\mathcal{K}_{f,P_i}|(|x_i| + |y_i|)}{\dfrac{1}{r+2}\left(1 - \left(\dfrac{r+1}{4}\right)|\mathcal{K}_{f,max}|\eta\right)\displaystyle\sum_i(\parallel \bar{x}_i \parallel^2)}$$

$$\leq \frac{\dfrac{1}{\sqrt{2}}\displaystyle\sum_i \parallel \bar{x}_i \parallel^2 |\mathcal{K}_{f,P_i}| \parallel \bar{x}_i \parallel}{\dfrac{1}{r+2}\left(1 - \left(\dfrac{r+1}{4}\right)|\mathcal{K}_{f,max}|\eta\right)\displaystyle\sum_i(\parallel \bar{x}_i \parallel^2)}$$

(since $\parallel \bar{x}_i \parallel_1 \leq \sqrt{2} \parallel \bar{x}_i \parallel_2$, for any 2-vector $\bar{x}_i$)

$$\leq \frac{\dfrac{1}{\sqrt{2}}\sqrt{|\mathcal{K}_{f,max}|\eta}\displaystyle\sum_i \parallel \bar{x}_i \parallel^2}{\dfrac{1}{r+2}\left(1 - \left(\dfrac{r+1}{4}\right)|\mathcal{K}_{f,max}|\eta\right)\displaystyle\sum_i(\parallel \bar{x}_i \parallel^2)}$$

$$= \frac{\dfrac{1}{\sqrt{2}}(r+2)L}{1 - \left(\dfrac{r+1}{4}\right)L^2}$$

$$= \frac{2\sqrt{2}(r+2)L}{4-(r+1)L^2} = B_r$$

where $L = \sqrt{|\mathcal{K}_{f,max}|\eta}$. It is similarly seen that

$$|u_{13}| \leq B_1.$$

Clearly for a given L, $B_r \geq B_1$. Thus we have the following.

$$\begin{aligned} |u_{13}|^2 + |u_{23}|^2 &\leq B_1^2 + B_r^2 \\ &\leq 2B_r^2 \end{aligned}$$

So for any $0 < \tau < 1$, if $2B_r^2 < \tau^2$ then we have the following.

$$\begin{aligned} \cos^2\theta &= 1 - (|u_{13}|^2 + |u_{23}|^2) \\ &\geq 1 - 2B_r^2 \\ &> 1 - \tau^2 \\ \Rightarrow |\theta| &< \cos^{-1}(\sqrt{1-\tau^2}) \end{aligned}$$

We now derive an upper bound on η so that this is guaranteed. We have the equivalence of the following conditions.

$$\begin{aligned} & 2B_r^2 < \tau^2 \\ \Leftrightarrow & \frac{2\sqrt{2}(r+2)L}{4-(r+1)L^2} < \frac{\tau}{\sqrt{2}} \\ \Leftrightarrow & \frac{\tau}{\sqrt{2}}(r+1)L^2 + 2\sqrt{2}(r+2)L - 2\sqrt{2}\tau < 0 \\ \Leftrightarrow & L < \frac{2G_{\tau,r}}{1+\sqrt{1+G_{\tau,r}^2(r+1)}} \\ & \left(\text{where} \quad G_{\tau,r} = \frac{\tau}{r+2}\right) \\ \Leftrightarrow & \eta < \left(\frac{H_{\tau,r}^2}{|\mathcal{K}_{f,max}|}\right) \end{aligned} \tag{2.24}$$

$$\text{where,} \qquad H_{\tau,r} = \left(\frac{2G_{\tau,r}}{1+\sqrt{1+G_{\tau,r}^2(r+1)}}\right).$$

Thus using (2.24), we arrive at the following bounds on the norms for the points in $\mathcal{N}_\varepsilon(P)$.

$$\text{As} \qquad \eta < \left(\frac{H_{\tau,r}^2}{|\mathcal{K}_{f,max}|}\right)$$

$$\text{We have} \qquad \|\,\bar{x}_{max,\mathcal{N}_\varepsilon(P)}\,\| < \frac{H_{\tau,r}}{|\mathcal{K}_{f,max}|}$$

Furthermore, as shown earlier in (2.5), we arrive at the following bound on the ambient space norm, which ensures that the above tangent space norm bound is guaranteed.

$$\varepsilon_{max,\mathcal{N}_\varepsilon(P)} \leq \sqrt{\left(\frac{\eta}{|\mathcal{K}_{f,max}|} + \frac{\eta^2|\mathcal{K}_{f,low}|^2}{4|\mathcal{K}_{f,max}|^2}\right)}$$

$$< \sqrt{\frac{H_{\tau,r}^2}{|\mathcal{K}_{f,max}|^2} + \frac{H_{\tau,r}^4|\mathcal{K}_{f,low}|^2}{4|\mathcal{K}_{f,max}|^4}}$$

$$= \left(\frac{H_{\tau,r}}{|\mathcal{K}_{f,max}|}\right)\sqrt{1 + \frac{H_{\tau,r}^2|\mathcal{K}_{f,low}|^2}{4|\mathcal{K}_{f,max}|^2}}$$

Conclusion: Let us consider that the points in $\mathcal{N}_\varepsilon(P)$ are sampled from within a disc of radius $\sqrt{\dfrac{\eta}{|\mathcal{K}_{f,max}|}}$ in T_PS, such that for any $r > 1$, $0 < \tau < 1$ the following conditions hold.

(i) $\eta < \dfrac{H_{\tau,r}^2}{|\mathcal{K}_{f,max}|}$ where

$$H_{\tau,r} = \frac{2G_{\tau,r}}{1 + \sqrt{1 + G_{\tau,r}^2(r+1)}},$$
$$G_{\tau,r} = \frac{\tau}{r+2}.$$

(ii) The rank of XX^T is at least 2, and $\dfrac{\lambda_1}{\lambda_2} < r$ ($\lambda_1 \geq \lambda_2 \geq \lambda_3$ are the eigenvalues of XX^T).

Then the modulus of the "angle" (as defined in [2]) between $\hat{T}_PS$ and T_PS will be upper bounded by $\cos^{-1}(\sqrt{1-\tau^2})$.

2.6 Random Sampling: Performance analysis of optimal locally estimated linear subspace

In the previous section we saw that an "angle" bound between $\hat{T}_P S$ and $T_P S$ is guaranteed if the points in $\mathcal{N}_\varepsilon(P)$ are sampled from within a disc (in $T_P S$) of sufficiently small radius. In particular we saw that in order to guarantee an arbitrarily low "angle" bound, the radius of the sampling disc would be arbitrarily close to 0. In this section we show that if the points in $\mathcal{N}_\varepsilon(P)$ are formed by sampling uniformly and independently at random in $T_P S$, from within a disc whose radius depends solely on the local principal curvature values $\mathcal{K}_{f,max}$ and $\mathcal{K}_{f,min}$, then as the number of samples $K \to \infty$, we have $\hat{T}_P S = T_P S$. We derive the expression for the bound on the radius of the sampling disc, and also derive a corresponding lower bound on K, so that if both bounds are satisfied then it gurantees probabilistically an upper bound on the "angle"[2] between $\hat{T}_P S$ and $T_P S$.

Let $\{x_i, y_i, f(x_i, y_i)\}_{i=1}^{K}$ denote K points from $\mathcal{N}_\varepsilon(P)$.

$$\text{Let } X^{(K)} = \begin{bmatrix} x_1 & x_2 & \cdots & x_K \\ y_1 & y_2 & \cdots & y_K \\ f(x_1, y_1) & f(x_2, y_2) & \cdots & f(x_K, y_K) \end{bmatrix}$$

Then,

$$\mathrm{C}^{(K)} = \frac{1}{K} X X^{T^{(K)}} = \begin{bmatrix} \frac{1}{K}\sum_i x_i^2 & \frac{1}{K}\sum_i x_i y_i & \frac{1}{K}\sum_i x_i f(x_i, y_i) \\ \frac{1}{K}\sum_i y_i x_i & \frac{1}{K}\sum_i y_i^2 & \frac{1}{K}\sum_i y_i f(x_i, y_i) \\ \frac{1}{K}\sum_i x_i f(x_i, y_i) & \frac{1}{K}\sum_i y_i f(x_i, y_i) & \frac{1}{K}\sum_i f^2(x_i, y_i) \end{bmatrix} = U \Lambda U^T.$$

Say each point is formed by sampling uniformly at random from within a disc of radius $\sqrt{\frac{\eta}{|\mathcal{K}_{f,max}|}}$ in $T_P S$. Representing the points in polar coordinates:

$$M_i \sim \mathrm{U}\left[0, \sqrt{\frac{\eta}{|\mathcal{K}_{f,max}|}}\right] \qquad \text{i.i.d}$$
$$\text{and,} \quad \theta_i \sim \mathrm{U}[0, 2\pi] \qquad \text{i.i.d} \quad \forall i = 1, \ldots, K$$

where $\{M_i\}_{i=1}^{K}$ and $\{\theta_i\}_{i=1}^{K}$ are independent random variables and U denotes the uniform distribution. Note that θ_i is the angle between $[x_i \ y_i]$ and $[1 \ 0]$. Let θ_{offset} denote the angle between $\bar{v}_1$ (the principal curvature direction corresponding to $\mathcal{K}_{f,max}$) and [1,0]. Thus we have,

$$\begin{aligned} f(x_i, y_i) &= \frac{1}{2} M_i^2 \mathcal{K}_{f,P_i} \\ x_i &= M_i \cos\theta_i \\ y_i &= M_i \sin\theta_i \end{aligned}$$

where $\mathcal{K}_{f,P_i} = \mathcal{K}_{f,max}\cos^2(\theta_i - \theta_{\text{offset}}) + \mathcal{K}_{f,min}\sin^2(\theta_i - \theta_{\text{offset}})$. So,

$$\mathrm{C}^{(K)} = \begin{bmatrix} \frac{1}{K}\sum_i M_i^2\cos^2\theta_i & \frac{1}{K}\sum_i M_i^2\cos\theta_i\sin\theta_i & \frac{1}{K}\sum_i \frac{1}{2}M_i^3\cos\theta_i\mathcal{K}_{f,P_i} \\ \frac{1}{K}\sum_i M_i^2\cos\theta_i\sin\theta_i & \frac{1}{K}\sum_i M_i^2\sin^2\theta_i & \frac{1}{K}\sum_i \frac{1}{2}M_i^3\sin\theta_i\mathcal{K}_{f,P_i} \\ \frac{1}{K}\sum_i \frac{1}{2}M_i^3\cos\theta_i\mathcal{K}_{f,P_i} & \frac{1}{K}\sum_i \frac{1}{2}M_i^3\sin\theta_i\mathcal{K}_{f,P_i} & \frac{1}{K}\sum_i \frac{1}{4}M_i^4\mathcal{K}_{f,P_i}^2 \end{bmatrix}.$$

It is easy to verify that

$$\begin{aligned} \mathbb{E}_{M,\theta}[M^2\cos^2\theta] &= \mathbb{E}_{M,\theta}[M^2\sin^2\theta] = \frac{\eta}{6|\mathcal{K}_{f,max}|}, \\ \text{and} \quad \mathbb{E}_{M,\theta}&[M^2\cos\theta\sin\theta] = 0. \end{aligned}$$

Observe that

$$\begin{aligned} \mathbb{E}_\theta[\mathcal{K}_{f,P}\cos\theta] &= \frac{1}{2\pi}\int_0^{2\pi}(\mathcal{K}_{f,max}\cos^2(\theta-\theta_{\text{offset}}) + \mathcal{K}_{f,min}\sin^2(\theta-\theta_{\text{offset}}))\cos\theta\ d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}(\mathcal{K}_{f,min} + (\mathcal{K}_{f,max} - \mathcal{K}_{f,min})\cos^2(\theta-\theta_{\text{offset}}))\cos\theta\ d\theta \\ &= 0 \qquad (\text{Since} \int_0^{2\pi}\cos^3\theta\ d\theta = 0) \end{aligned}$$

Similary, $\mathbb{E}_\theta[\mathcal{K}_{f,P}\sin\theta] = 0$. Thus,

$$\mathbb{E}_{M,\theta}\left[\frac{1}{2}M^3\cos\theta\mathcal{K}_{f,P}\right] = \mathbb{E}_{M,\theta}\left[\frac{1}{2}M^3\sin\theta\mathcal{K}_{f,P}\right] = 0$$

Furthermore,

$$\begin{aligned} \mathbb{E}_\theta\left[\frac{1}{4}\mathcal{K}_{f,P}^2\right] &= \mathbb{E}_\theta\left[\frac{1}{4}(\mathcal{K}_{f,max}\cos^2(\theta-\theta_{\text{offset}}) + \mathcal{K}_{f,min}\sin^2(\theta-\theta_{\text{offset}}))^2\right] \\ &= \frac{1}{4}\mathbb{E}_\theta[\mathcal{K}_{f,max}^2\cos^4(\theta+\theta_{\text{offset}}) + \mathcal{K}_{f,min}^2\sin^4(\theta-\theta_{\text{offset}}) + \end{aligned}$$
$$\frac{1}{2}\mathcal{K}_{f,max}\mathcal{K}_{f,min}\sin^2(2(\theta-\theta_{\text{offset}}))]$$

$$\text{Now,} \qquad \mathbb{E}_\theta[\cos^4(\theta - \theta_{\text{offset}})] = \mathbb{E}_\theta[\sin^4(\theta - \theta_{\text{offset}})] = \frac{3}{8}$$

$$\mathbb{E}_\theta\left[\frac{1}{2}\sin^2(2(\theta - \theta_{\text{offset}}))\right] = \frac{1}{2}\mathbb{E}_\theta\left[\frac{1 - \cos 4(\theta - \theta_{\text{offset}})}{2}\right] = \frac{1}{4}$$

$$\begin{aligned}
\therefore \qquad \mathbb{E}_\theta\left[\frac{1}{4}\mathcal{K}_{f,P}^2\right] &= \frac{1}{32}[3(\mathcal{K}_{f,max}^2 + \mathcal{K}_{f,min}^2) + 2\mathcal{K}_{f,max}\mathcal{K}_{f,min}] \\
&= \frac{1}{32}[3(|\mathcal{K}_{f,max}|^2 + |\mathcal{K}_{f,min}|^2) \pm 2|\mathcal{K}_{f,max}||\mathcal{K}_{f,min}|]
\end{aligned}$$

(where '+' denotes $\mathcal{K}_{f,max}\mathcal{K}_{f,min} \geq 0$ and '-' denotes $\mathcal{K}_{f,max}\mathcal{K}_{f,min} \leq 0$)

$$\text{Thus,} \qquad \mathbb{E}_{M,\theta}\left[\frac{1}{4}M_i^4\mathcal{K}_{f,P}^2\right] = \frac{\eta^2}{160}\left(3 + 3\frac{|\mathcal{K}_{f,min}|^2}{|\mathcal{K}_{f,max}|^2} \pm 2\frac{|\mathcal{K}_{f,min}|}{|\mathcal{K}_{f,max}|}\right)$$

As $K \to \infty$, then by the Strong Law of Large numbers, $[C^{(K)}]_{i,j}$ converges a.s to $[C]_{i,j}$ $(1 \leq i,j \leq 3)$, where

$$\text{C} = \begin{bmatrix} \frac{\eta}{6|\mathcal{K}_{f,max}|} & 0 & 0 \\ 0 & \frac{\eta}{6|\mathcal{K}_{f,max}|} & 0 \\ 0 & 0 & \frac{\eta^2}{160}\left(3 + 3\frac{|\mathcal{K}_{f,min}|^2}{|\mathcal{K}_{f,max}|^2} \pm 2\frac{|\mathcal{K}_{f,min}|}{|\mathcal{K}_{f,max}|}\right) \end{bmatrix}.$$

Now the optimal 2-dimensional linear subpace (in the least squares sense), $\hat{T}_P(S)$, passing through the local origin P, would be the span of the 2 eigenvectors corresponding to the 2 largest eigenvalues of C. Now $[1,\ 0,\ 0]^T$ and $[0,\ 1,\ 0]^T$ correspond to the eigenvalue $\frac{\eta}{6|\mathcal{K}_{f,max}|}$. Furthermore, the actual tangent space T_PS is span$\{[1,\ 0,\ 0]^T, [0,\ 1,\ 0]^T\}$. Therefore $\hat{T}_P(S)$ would be exactly T_PS as $K \to \infty$ if

$$\frac{\eta}{6|\mathcal{K}_{f,max}|} > \frac{\eta^2}{160}\left(3 + 3\frac{|\mathcal{K}_{f,min}|^2}{|\mathcal{K}_{f,max}|^2} \pm 2\frac{|\mathcal{K}_{f,min}|}{|\mathcal{K}_{f,max}|}\right),$$

$$\begin{aligned}
\text{or} \quad \eta &< \frac{80}{3|\mathcal{K}_{f,max}|\left(3 + 3\frac{|\mathcal{K}_{f,min}|^2}{|\mathcal{K}_{f,max}|^2} \pm 2\frac{|\mathcal{K}_{f,min}|}{|\mathcal{K}_{f,max}|}\right)} \\
&= \eta_{\text{bound}}.
\end{aligned}$$

Now each entry in $C^{(K)}$ is a sum of K i.i.d bounded random variables, where the random variables involved have the following bounds.

$$
\begin{aligned}
0 \;&\leq\; M^2 \cos^2\theta \;\leq\; \frac{\eta}{|\mathcal{K}_{f,max}|} \\
0 \;&\leq\; M^2 \sin^2\theta \;\leq\; \frac{\eta}{|\mathcal{K}_{f,max}|} \\
\frac{-\eta}{2|\mathcal{K}_{f,max}|} \;&\leq\; M^2 \cos\theta \sin\theta \;\leq\; \frac{\eta}{2|\mathcal{K}_{f,max}|} \\
0 \;&\leq\; \frac{1}{4} M^4 \mathcal{K}_{f,P}^2 \;\leq\; \frac{\eta^2}{4} \\
\frac{-\eta^{\frac{3}{2}}}{2\sqrt{|\mathcal{K}_{f,max}|}} \;&\leq\; \frac{1}{2} M^3 \sin\theta\; \mathcal{K}_{f,P} \;\leq\; \frac{\eta^{\frac{3}{2}}}{2\sqrt{|\mathcal{K}_{f,max}|}} \\
\frac{-\eta^{\frac{3}{2}}}{2\sqrt{|\mathcal{K}_{f,max}|}} \;&\leq\; \frac{1}{2} M^3 \cos\theta\; \mathcal{K}_{f,P} \;\leq\; \frac{\eta^{\frac{3}{2}}}{2\sqrt{|\mathcal{K}_{f,max}|}}
\end{aligned}
$$

We would now like to find a lower bound on K which if satisfied ensures with a probability of at least $1-\beta$ (where $0 < \beta < 1$) individually, for each entry of $C^{(K)}$ that it will lie within an ϵ interval of its expected value. Given this, it is then easy to verify using the union bound that the probability of the event where all entries of $C^{(K)}$ lie within an ϵ-interval of their expected value is at least $1-6\beta$.

We proceed by using Hoeffding's Inequality which states the following. Given K bounded i.i.d random variables $\{Y_i\}_{i=1}^{K}$, where $a_1 \leq Y_i \leq a_2$ a.s ($a_1 < a_2$), then for any $\epsilon > 0$,

$$
\mathbb{P}\left(\left| \frac{1}{K} \sum_{i=1}^{K} Y_i - \mathbb{E}[Y_i] \right| \;\geq\; \epsilon \right) \leq 2e^{\dfrac{-2K\epsilon^2}{(a_2 - a_1)^2}}
$$

We now analyze each term of $C^{(K)}$ separately.

$$(1) \qquad \mathbb{P}\left(\left|\frac{1}{K}\sum_{i=1}^{K} M_i^2\cos^2\theta_i - \frac{\eta}{6|\mathcal{K}_{f,max}|}\right| \;\geq\; \epsilon\right) \;\leq\; 2e^{\dfrac{-2K\epsilon^2}{\left(\eta^2/|\mathcal{K}_{f,max}|^2\right)}}$$

$$= 2e^{\dfrac{-2K\epsilon^2|\mathcal{K}_{f,max}|^2}{\eta^2}}.$$

$$\text{Thus,} \qquad 2e^{\dfrac{-2K\epsilon^2|\mathcal{K}_{f,max}|^2}{\eta^2}} < \beta < 1,$$

$$\Leftrightarrow K > \frac{ln(\frac{2}{\beta})\eta^2}{2\epsilon^2|\mathcal{K}_{f,max}|^2} = K^{(1)}_{\text{bound}}.$$

Observe that the above condition on K applies for $\frac{1}{K}\sum_i M_i^2\cos\theta_i\sin\theta_i$ and $\frac{1}{K}\sum_i M_i^2\sin^2\theta_i$ too since the difference between the upper and lower bounds for the random variables involved in the summation is the same.

$$(2) \qquad \mathbb{P}\left(\left|\frac{1}{K}\sum_{i=1}^{K} M_i^4\mathcal{K}_{f,P_i}^2 - \mathbb{E}\left[\frac{1}{4}M_i^4\mathcal{K}_{f,P_i}^2\right]\right| \;\geq\; \epsilon\right) \;\leq\; 2e^{\dfrac{-2K\epsilon^2}{(\eta^4/16)}}$$

$$= 2e^{\dfrac{-32K\epsilon^2}{\eta^4}}$$

$$\text{Thus,} \qquad 2e^{\dfrac{-32K\epsilon^2}{\eta^4}} < \beta < 1,$$

$$\Leftrightarrow K > \frac{ln(\frac{2}{\beta})\eta^4}{32\epsilon^2} = K^{(2)}_{\text{bound}}.$$

$$(3) \qquad \mathbb{P}\left(\left|\frac{1}{K}\sum_{i=1}^{K} \frac{1}{2}M_i^3\cos\theta_i\mathcal{K}_{f,P_i} - 0\right| \;\geq\; \epsilon\right) \;\leq\; 2e^{\dfrac{-2K\epsilon^2}{(\eta^3/|\mathcal{K}_{f,max}|)}}$$

$$= 2e^{\dfrac{-2K\epsilon^2|\mathcal{K}_{f,max}|}{\eta^3}}$$

$$\text{Thus,} \qquad 2e^{\dfrac{-2K\epsilon^2|\mathcal{K}_{f,max}|}{\eta^3}} < \beta < 1,$$

$$\Leftrightarrow K > \frac{ln(\frac{2}{\beta})\eta^3}{2\epsilon^2|\mathcal{K}_{f,max}|} = K^{(3)}_{\text{bound}}.$$

Note that the above condition on K is the same for $\frac{1}{K}\sum_{i=1}^{K}\frac{1}{2}M_i^3\sin\theta_i\mathcal{K}_{f,P_i}$, for the same reason explained earlier. So we essentially have 3 lower bounds namely $K_{\text{bound}}^{(1)}$, $K_{\text{bound}}^{(2)}$ and $K_{\text{bound}}^{(3)}$. Observe that

$$\begin{aligned} K_{\text{bound}}^{(1)} &> K_{\text{bound}}^{(2)} \quad \text{and} \quad K_{\text{bound}}^{(1)} > K_{\text{bound}}^{(3)} \\ \Leftrightarrow \eta &< \frac{1}{|\mathcal{K}_{f,max}|}; \end{aligned}$$

$$\begin{aligned} K_{\text{bound}}^{(2)} &> K_{\text{bound}}^{(1)} \quad \text{and} \quad K_{\text{bound}}^{(2)} > K_{\text{bound}}^{(3)} \\ \Leftrightarrow \eta &> \frac{16}{|\mathcal{K}_{f,max}|}; \end{aligned}$$

$$\begin{aligned} K_{\text{bound}}^{(3)} &> K_{\text{bound}}^{(1)} \quad \text{and} \quad K_{\text{bound}}^{(3)} > K_{\text{bound}}^{(2)} \\ \Leftrightarrow \frac{1}{|\mathcal{K}_{f,max}|} &< \eta < \frac{16}{|\mathcal{K}_{f,max}|}. \end{aligned}$$

Hence for any $\epsilon > 0$, $0 < \beta < 1$ we conclude the following.

(i) If $\eta < \frac{1}{|\mathcal{K}_{f,max}|}$, provided that $K > \frac{ln(\frac{2}{\beta})\eta^2}{2\epsilon^2|\mathcal{K}_{f,max}|^2}$, then with probabilty at least $(1-6\beta)$ no term of $C^{(K)}$ will deviate from its expected value by more than ϵ.

(ii) If $\frac{1}{|\mathcal{K}_{f,max}|} < \eta < \frac{16}{|\mathcal{K}_{f,max}|}$, provided that $K > \frac{ln(\frac{2}{\beta})\eta^3}{2\epsilon^2|\mathcal{K}_{f,max}|}$, then with probabilty at least $(1-6\beta)$ no term of $C^{(K)}$ will deviate from its expected value by more than ϵ.

(iii) If $\eta > \frac{16}{|\mathcal{K}_{f,max}|}$, provided that $K > \frac{ln(\frac{2}{\beta})\eta^4}{32\epsilon^2}$, then with probabilty at least $(1-6\beta)$ no term of $C^{(K)}$ will deviate from its expected value by more than ϵ.

As shown earlier, if $\eta < \eta_{\text{bound}}$ then as $K\to\infty$, $\hat{T}_PS = T_PS$. We are interested in upper bounding the "angle"[2] between $\hat{T}_PS$ and T_PS. Hence for further analysis we assume that $\eta < \eta_{\text{bound}}$. We now analyze perturbation of C.

Let $C' = C + \triangle$,

$$C = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} \qquad \triangle = \begin{bmatrix} \triangle_{11} & \triangle_{12} & \triangle_{13} \\ \triangle_{12} & \triangle_{22} & \triangle_{23} \\ \triangle_{13} & \triangle_{23} & \triangle_{33} \end{bmatrix}$$

where,

$$\begin{aligned} a &= \frac{\eta}{6|\mathcal{K}_{f,max}|}, \\ b &= \frac{\eta^2}{160}\left(3 + 3\frac{|\mathcal{K}_{f,min}|^2}{|\mathcal{K}_{f,max}|^2} \pm 2\frac{|\mathcal{K}_{f,min}|}{|\mathcal{K}_{f,max}|}\right). \end{aligned}$$

Note that $a > b$ as $\eta < \eta_{\text{bound}}$. Say $|\triangle_{ij}| < \epsilon$ for some $\epsilon > 0$. We seek to find an upper bound on ϵ to guarantee an upper bound on the angle between $T_P S$ and $\hat{T}_P S$. Consider the following.

$$\mathcal{C}_\epsilon = [C - \epsilon E, C + \epsilon E] \quad \text{(notation of interval matrices)}$$

$$\text{where} \quad E = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$\mathcal{C}_\epsilon$ denotes the set of matrices C' s.t $[C']_{i,j} \in [[C]_{i,j} - \epsilon, [C]_{i,j} + \epsilon] \; \forall \; i, j = 1, 2, 3$. Let $\lambda_1(C') \geq \lambda_2(C') \geq \lambda_3(C')$ denote the eigenvalues of any $C' \in \mathcal{C}_\epsilon$. Now from [3] we know that

$$\lambda_i(C) - \rho(\epsilon E) \leq \lambda_i(C') \leq \lambda_i(C) + \rho(\epsilon E) \; (\forall \; i = 1,2,3)$$

where $\rho(A)$ denotes the spectral radius of A. Therefore we have the following bounds on the eigenvalues of any $C' \in \mathcal{C}_\epsilon$.

$$\begin{aligned} a - 3\epsilon \leq \lambda_1(C') \leq a + 3\epsilon \\ a - 3\epsilon \leq \lambda_2(C') \leq a + 3\epsilon \\ b - 3\epsilon \leq \lambda_3(C') \leq b + 3\epsilon \end{aligned}$$

Consider,

$$\begin{aligned} C'\bar{u}_2 &= \lambda_2(C')\bar{u}_2 \\ \Rightarrow \triangle_{13}u_{21} + \triangle_{23}u_{22} + (b + \triangle_{33})u_{23} &= \lambda_2 u_{23} \\ \Rightarrow \lambda_2|u_{23}| &\leq 2\epsilon + (b+\epsilon)|u_{23}| \\ \text{or} \qquad (\lambda_2 - b - \epsilon)|u_{23}| &\leq 2\epsilon \end{aligned}$$

Note that,

$$\lambda_2 \;\geq\; a - 3\epsilon$$
$$\Rightarrow \lambda_2 - b - \epsilon \;\geq\; a - b - 4\epsilon$$

$\therefore\ \epsilon \;<\; \left(\frac{a-b}{4}\right)$ ensures $\lambda_2 - b - \epsilon \;>\; 0$ (and thus $\lambda_1 - b - \epsilon \;>\; 0$). So assuming ϵ satisfies this condition we have

$$|u_{13}|, |u_{23}| \;<\; \frac{2\epsilon}{a - b - 4\epsilon} \;=\; B_\epsilon$$

Proceeding as in Section 2.5, we see that if $2B_\epsilon^2 \;<\; \tau^2$ $(0 < \tau < 1)$, then

$$\begin{aligned} \cos^2\theta &= 1 - (|u_{13}|^2 + |u_{23}|^2) \\ &\geq 1 - 2B_\epsilon^2 \\ &> 1 - \tau^2 \\ \text{or} \quad |\theta| &< \cos^{-1}(\sqrt{1-\tau^2}) \end{aligned}$$

$$\therefore B_\epsilon \;<\; \frac{\tau}{\sqrt{2}}$$
$$\Leftrightarrow \epsilon \;<\; \frac{(a-b)\frac{\tau}{\sqrt{2}}}{2 + 2\sqrt{2}\tau}$$

Note that $\dfrac{(a-b)\frac{\tau}{\sqrt{2}}}{2 + 2\sqrt{2}\tau} \;<\; \left(\frac{a-b}{4}\right)$. Lastly, observe the following for η_{bound}.

$$\eta_{\text{bound}} = \frac{80}{3|\mathcal{K}_{f,max}|\left(3 + 3\dfrac{|\mathcal{K}_{f,min}|^2}{|\mathcal{K}_{f,max}|^2} \pm 2\dfrac{|\mathcal{K}_{f,min}|}{|\mathcal{K}_{f,max}|}\right)}$$

$$\begin{aligned} \therefore \qquad & \frac{10}{3|\mathcal{K}_{f,max}|} < \eta_{\text{bound}} < \frac{80}{9|\mathcal{K}_{f,max}|}; \qquad \text{if} \qquad \mathcal{K}_{f,max}\mathcal{K}_{f,min} \geq 0 \\ \text{and,} \qquad & \frac{20}{3|\mathcal{K}_{f,max}|} < \eta_{\text{bound}} < \frac{10}{|\mathcal{K}_{f,max}|}; \qquad \text{if} \qquad \mathcal{K}_{f,max}\mathcal{K}_{f,min} \leq 0 \end{aligned}$$

Conclusion: If we have K points in $\mathcal{N}_\varepsilon(P)$ which are sampled independently and uniformly at random from a disc of radius $\sqrt{\dfrac{\eta}{|\mathcal{K}_{f,max}|}}$ in T_PS, then for any $\eta \;<\; \eta_{\text{bound}}$, $0 \;< \epsilon \;<\; \dfrac{(a-b)\frac{\tau}{\sqrt{2}}}{2 + 2\sqrt{2}\tau}$, $0 \;<\; \beta \;<\; 1$ and $0 \;<\; \tau \;<\; 1$

where $a = \frac{\eta}{6|\mathcal{K}_{f,max}|}$, $b = \frac{\eta^2}{160}\left(3 + 3\frac{|\mathcal{K}_{f,min}|^2}{|\mathcal{K}_{f,max}|^2} \pm 2\frac{|\mathcal{K}_{f,min}|}{|\mathcal{K}_{f,max}|}\right)$ we conclude the following.

(i) If $\eta < \frac{1}{|\mathcal{K}_{f,max}|}$, provided that $K > \frac{ln(\frac{2}{\beta})\eta^2}{2\epsilon^2|\mathcal{K}_{f,max}|^2}$, then with probabilty at least $(1-6\beta)$ we have $|\theta| < \cos^{-1}\sqrt{1-\tau^2} = |\theta|_{\text{bound}}$.

(ii) If $\frac{1}{|\mathcal{K}_{f,max}|} < \eta < \eta_{\text{bound}}$, provided that $K > \frac{ln(\frac{2}{\beta})\eta^3}{2\epsilon^2|\mathcal{K}_{f,max}|}$, then with probabilty at least $(1-6\beta)$ we have $|\theta| < \cos^{-1}\sqrt{1-\tau^2} = |\theta|_{\text{bound}}$.

2.7 Experiment results: Optimal locally estimated linear subspace

In this section we present some simulation results for the sampling conditions on points, to guarantee an upper bound on the "angle", (θ) between $\hat{T}_P S$ and $T_P S$ as derived in the previous sections. We consider two surfaces, the first one with principal curvatures $\mathcal{K}_{f,max}$= 3, $\mathcal{K}_{f,min}$= -1 and the second one with principal curvatures $\mathcal{K}_{f,max}$= 3, $\mathcal{K}_{f,min}$= 1. The angle bound parameter, $\tau = 0.1$, which means that $|\theta|_{\text{bound}} = 5.73°$. The value of $|\theta|$ was averaged over 50 trials in all simulations below.

2.7.1 Non Probabilistic Bounds from Section 2.5

The value of η was chosen to be : $0.95\frac{H^2_{\tau,r}}{|\mathcal{K}_{f,max}|}$. We see in Fig. 2.9 and Fig. 2.10 that $|\theta|$ is considerably lower than $|\theta|_{\text{bound}}$. Further note that since $\eta < \eta_{\text{bound}}$, then as was shown in Section 2.6, $|\theta|$ expectedly decreases with increase in K.

2.7.2 Probabilistic Bounds from Section 2.6

In this experiment we vary the value of η from $0.1\frac{1}{|\mathcal{K}_{f,max}|}$ to $0.5\eta_{\text{bound}}$ and observe the behaviour of $|\theta|$. For each value of η, we choose $\epsilon = 0.95\frac{(a-b)\frac{\tau}{\sqrt{2}}}{2+2\sqrt{2}\tau}$, where a and b are as defined in Section 2.6. $\beta = 0.1$ was chosen. For each value of η (and thus ϵ), a value of K satisfying the corresponding lower bound

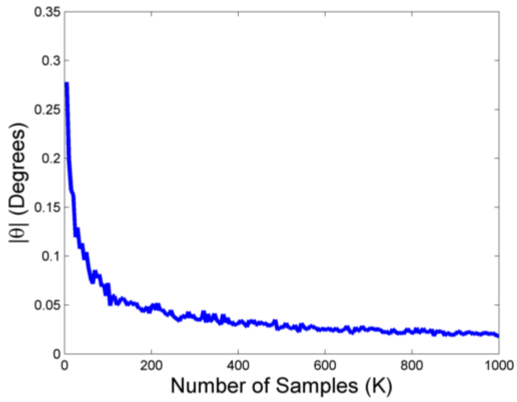

Figure 2.9: $\mathcal{K}_{f,max} = 3$, $\mathcal{K}_{f,min} = -1$. r = 3 was chosen and thus $\eta = 0.000133$. The number of samples varies from 5 to 1000. $|\theta|_{\text{min}} = 0.018°$ and $|\theta|_{\text{max}} = 0.278°$. $\eta_{\text{bound}} = 3.333$.

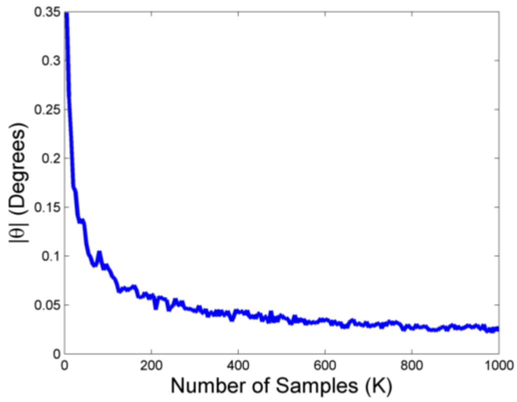

Figure 2.10: $\mathcal{K}_{f,max} = 3$, $\mathcal{K}_{f,min} = 1$. r = 3 was chosen and thus $\eta = 0.000133$. The number of samples varies from 5 to 1000. $|\theta|_{\text{min}} = 0.021°$ and $|\theta|_{\text{max}} = 0.349°$. $\eta_{\text{bound}} = 2.222$.

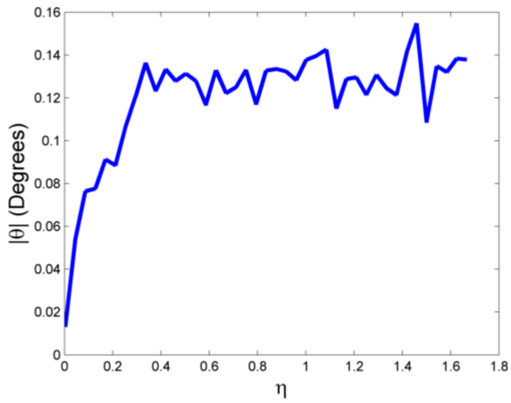

Figure 2.11: $\mathcal{K}_{f,max} = 3$, $\mathcal{K}_{f,min} = -1$. $|\theta|_{\text{min}} = 0.014°$ and $|\theta|_{\text{max}} = 0.148°$. $\eta_{\text{bound}} = 3.333$.

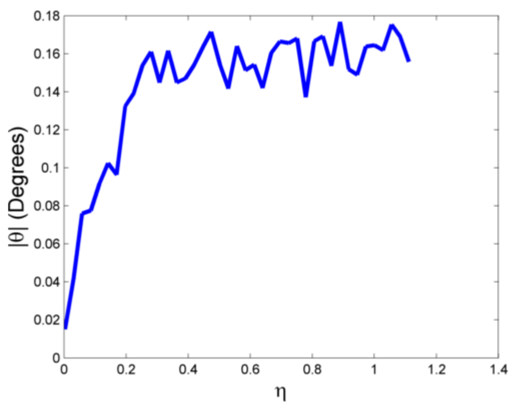

Figure 2.12: $\mathcal{K}_{f,max} = 3$, $\mathcal{K}_{f,min} = 1$. $|\theta|_{\text{min}} = 0.016°$ and $|\theta|_{\text{max}} = 0.178°$. $\eta_{\text{bound}} = 2.222$.

(as explained in Section 2.6)was chosen. We see in Fig. 2.11 and Fig. 2.12, that $|\theta|$ initially increases as η (or in other words the sampling radius) increases and then later fluctuates around a certain value. Observe that $|\theta|$ is considerably lower than $|\theta|_{\text{bound}}$.

2.7.3 Significance of η_{bound}

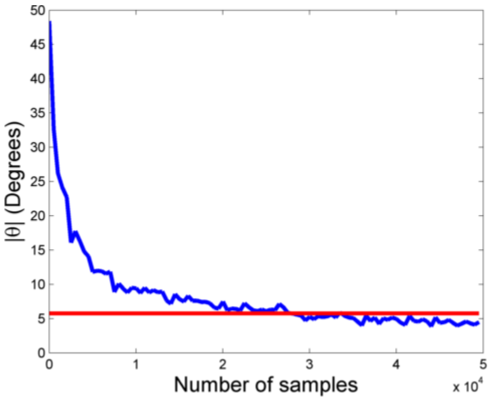

Figure 2.13: $\mathcal{K}_{f,max} = 3, \mathcal{K}_{f,min} = -1$. $|\theta|_{\min} = 3.88°$ and $|\theta|_{\max} = 48.42°$. $\eta = 0.9\eta_{\text{bound}}$. Number of samples varied from 10 to 50,000.

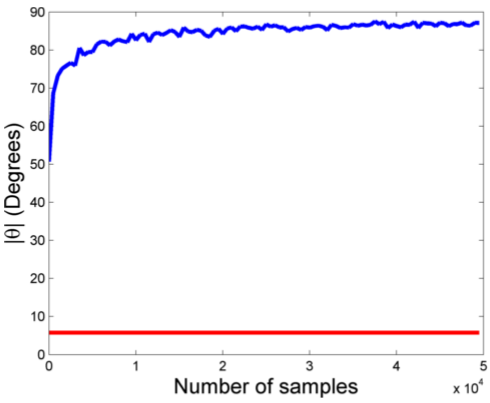

Figure 2.14: $\mathcal{K}_{f,max} = 3$, $\mathcal{K}_{f,min} = -1$. $|\theta|_{\min} = 50.72°$ and $|\theta|_{\max} = 87.55°$. $\eta = 1.1\eta_{\text{bound}}$. Number of samples varied from 10 to 50,000.

From Fig. 2.13 we see that if η is slightly less than η_{bound}, then $|\theta|$ reduces as the number of samples increases and eventually becomes less than $|\theta|_{\text{bound}}$. Furthermore as is seen in Fig. 2.14 if η is slightly greater than η_{bound} than $|\theta|$ increases sharply as the number of samples increases and approaches 90°.

Chapter 3

Quadratic Embeddings of m-dimensional Riemannian manifolds in $\mathbb{R}^n$

In this chapter we consider quadratic embeddings of m-dimensional Riemannian manifolds in $\mathbb{R}^n$. The embedding is assumed to be quadratic at a reference point P. We derive sampling conditions for points lying in $\mathcal{N}_\varepsilon(P)$ such that they guarantee the specified performance criterion. In Section 3.1 we consider the trivial case where there is a single point in $\mathcal{N}_\varepsilon(P)$, and derive bounds on its norm such that a reconstruction error criterion is satisfied. In Section 3.2 we consider the case where $\mathcal{N}_\varepsilon(P)$ contains more than one point. As before we partition $\mathcal{N}_\varepsilon(P)$ into two disjoint subsets, namely S_1 and S_2, representing the local "high" curvature and "low" curvature regions respectively. We derive bounds on the norms of points lying in these sets so that a local reconstruction error criterion is satisfied. We omit the random sampling framework for local reconstruction error analysis due to its similarity with the results derived in Section 2.3.
In Section 3.3 we derive conditions for points in $\mathcal{N}_\varepsilon(P)$ so that the "angle" between the T_PS and $\hat{T}_PS$ is upper bounded. Finally Section 3.4 contains results for the case where the points are sampled uniformly at random. We derive bounds on the norm of the points and also a lower bound on the number of samples K so that if both of these conditions are met, then they guarantee probabilistically an upper bound on the "angle" between T_PS and $\hat{T}_PS$.

Let $M = [x_1 \ \dots \ x_m \ f_1(x_1, \dots, x_m) \ \dots \ f_{n-m}(x_1, \dots, x_m)]$ be a point in $\mathcal{N}_\varepsilon(P)$. Let $\bar{x}_M = [x_1 \ \dots \ x_m]$ denote the orthogonal projection of M on T_PS. The manifold S can be represented in terms of $n-m$ hypersurfaces, where the

i^{th} hypersurface $\mathcal{S}_i$ is given by $\{[x_1 \ldots x_m \; f_i(\bar{x})] : [x_1 \ldots x_m] \in T_P S\} \subset \mathbb{R}^{m+1}$. As the embedding is quadratic we have the following $\forall\, i = 1, \ldots, n-m$.

$$\begin{aligned} f_i(\bar{x}_M) &= f_i(\bar{0}) + \bigtriangledown_{f_i}(\bar{0})^T \bar{x}_M + \frac{1}{2}\bar{x}_M^T H_{f_i}(\bar{0})\bar{x}_M \\ &= \bar{0} + \bar{0}^T \bar{x}_M + \frac{1}{2}\bar{x}_M^T V_i \Lambda_i V_i^T \bar{x}_M \\ &= \frac{1}{2}\sum_{j=1}^{m}\left(< \bar{x}_M, \bar{v}_{i,j} >^2 \mathcal{K}_{f_i,j}\right) \\ &= \frac{1}{2} \parallel \bar{x}_M \parallel^2 \sum_{j=1}^{m}(r_{i,j,M}\mathcal{K}_{f_i,j}) \\ &= \frac{1}{2} \parallel \bar{x}_M \parallel^2 \mathcal{K}_{f_i,M} \end{aligned}$$

where $r_{i,j,M} = \dfrac{< \bar{x}_M, \bar{v}_{i,j} >^2}{\parallel \bar{x}_M \parallel^2}$ so that $\displaystyle\sum_{j=1}^{m}(r_{i,j,M}) = 1$ and $r_{i,j,M} \geq 0 \;\forall$ i,j.
Geometrically $\mathcal{K}_{f_i,M} = \sum_{j=1}^{m}(r_{i,j,M}\mathcal{K}_{f_i,j})$ represents the curvature of the geodesic curve from P to M, for the i^{th} hypersurface $\mathcal{S}_i$.

$$V_i = \begin{bmatrix}\bar{v}_{i,1} & \bar{v}_{i,2} & \cdots & \bar{v}_{i,m}\end{bmatrix}_{m\times m}, \qquad \Lambda_i = \begin{bmatrix}\mathcal{K}_{f_i,1} & & & \\ & \mathcal{K}_{f_i,2} & & \\ & & \ddots & \\ & & & \mathcal{K}_{f_i,m}\end{bmatrix}_{m\times m}$$

denote respectively the eigenvector and eigenvalue matrices of $H_{f_i}(\bar{0})$. Say $\forall\, i \in \{1, \ldots, n-m\}$

$$\begin{aligned} |\mathcal{K}_{f_i,max}| &= \max_{j=1,2,\ldots,m}|\mathcal{K}_{f_i,j}|, \\ \text{and} \quad |\mathcal{K}_{f_i,min}| &= \min_{j=1,2,\ldots,m}|\mathcal{K}_{f_i,j}|. \end{aligned}$$

Then observe that

$$|\mathcal{K}_{f_i,M}| \in [|\mathcal{K}_{f_i,min}|, |\mathcal{K}_{f_i,max}|] \quad \text{if all elements of } \{\mathcal{K}_{f_i,j}\}_{j=1}^{m} \text{ are of the same sign.} \tag{3.1}$$

$$|\mathcal{K}_{f_i,M}| \in [0, |\mathcal{K}_{f_i,max}|] \quad \text{if } \exists \text{ at least 2 elements of } \{\mathcal{K}_{f_i,j}\}_{j=1}^{m} \text{ which are of opposite signs.} \tag{3.2}$$

So $|\mathcal{K}_{f_i,m}| \in [|\mathcal{K}_{f_i,low}|, |\mathcal{K}_{f_i,max}|]$, where $\mathcal{K}_{f_i,low} = |\mathcal{K}_{f_i,min}|$ if (3.1) is satisfied and $\mathcal{K}_{f_i,low} = 0$ if (3.2) is satisfied.
Let $|\mathcal{K}_{f,max}| = \max_{i=1,\ldots,n-m} |\mathcal{K}_{f_i,max}|$. Assume for further analysis that $|\mathcal{K}_{f,max}| > 0$.

3.1 Performance analysis of local reconstruction error for single neighbouring sample

Let $E_M^2 = \sum_{i=1}^{n-m} |f_i(\bar{x}_M)|^2$ denote the reconstruction error for M.

$$\text{Hence} \quad E_M^2 \leq \frac{\gamma^2}{4} \Leftrightarrow \| \bar{x}_M \|^4 \sum_{i=1}^{n-m} |\mathcal{K}_{f_i,M}|^2 \leq \gamma^2. \tag{3.3}$$

$$\text{But} \quad \| \bar{x}_M \|^4 \sum_{i=1}^{n-m} |\mathcal{K}_{f_i,M}|^2 \leq \| \bar{x}_M \|^4 \sum_{i=1}^{n-m} |\mathcal{K}_{f_i,max}|^2.$$

$$\text{Thus, if} \quad \| \bar{x}_M \|^4 \sum_{i=1}^{n-m} |\mathcal{K}_{f_i,max}|^2 \leq \gamma^2$$

$$\text{then} \quad \| \bar{x}_M \| \leq \sqrt[4]{\frac{\gamma^2}{\sum_{i=1}^{n-m} |\mathcal{K}_{f_i,max}|^2}}. \tag{3.4}$$

Any point which satisfies the above condition will also satisfy (3.3). Proceeding similarly as for the 2-dimensional case, we arrive at the following condition on the ambient space norm of any point M, which if satisfied ensures that (3.4) is satisfied.

$$\begin{aligned} \varepsilon_M^2 &\leq \frac{\gamma}{\sqrt{\sum_{i=1}^{n-m} |\mathcal{K}_{f_i,max}|^2}} + \frac{\gamma^2 \sum_{i=1}^{n-m} |\mathcal{K}_{f_i,low}|^2}{4 \sum_{i=1}^{n-m} |\mathcal{K}_{f_i,max}|^2} \\ \Leftrightarrow \varepsilon_M &\leq \sqrt{\frac{\gamma}{\sqrt{\sum_{i=1}^{n-m} |\mathcal{K}_{f_i,max}|^2}} + \frac{\gamma^2 \sum_{i=1}^{n-m} |\mathcal{K}_{f_i,low}|^2}{4 \sum_{i=1}^{n-m} |\mathcal{K}_{f_i,max}|^2}} \end{aligned} \tag{3.5}$$

Discussion: Note that if M is constrained to lie in a region such that

$$\sum_{i=1}^{n-m} |\mathcal{K}_{f_i,M}|^2 \in \left[\sum_{i=1}^{n-m} |\mathcal{K}_{f_i,low}|^2 \,,\, \sum_{i=1}^{n-m} \alpha_i^2 \right]$$

where $\alpha_i \in [|\mathcal{K}_{f_i,low}| \,,\, |\mathcal{K}_{f_i,max}|] \;;\; \forall\; i = 1, ..., n-m$, then $\| \bar{x}_M \| \leq \sqrt[4]{\frac{\gamma^2}{\sum_{i=1}^{n-m} \alpha_i^2}}$ ensures that (3.3) is satisfied. Furthermore, in this case the following condition on the ambient space norm of M

$$\varepsilon_M^2 \leq \frac{\gamma}{\sqrt{\sum_{i=1}^{n-m} \alpha_i^2}} + \frac{\gamma^2 \sum_{i=1}^{n-m} |\mathcal{K}_{f_i,low}|^2}{4 \sum_{i=1}^{n-m} \alpha_i^2}$$

$$\text{implies that} \quad \| \bar{x}_M \| \leq \sqrt[4]{\frac{\gamma^2}{\sum_{i=1}^{n-m} \alpha_i^2}}$$

3.2 Performance analysis of local reconstruction error: K points case

We now consider the case where $\mathcal{N}_\varepsilon(P) = \{P_1, ..., P_K\}$, $K > 1$.
Let P_i be $[x_1^{(i)}\ x_2^{(i)}\ \ldots\ x_m^{(i)}\ f_1(\bar{x}_{P_i})\ \ldots\ f_{n-m}(\bar{x}_{P_i})]$ where $\bar{x}_{P_i} = [x_1^{(i)}\ x_2^{(i)}\ \ldots\ x_m^{(i)}]$ denotes the orthogonal projection of P_i on $T_P S$.
Thus, $E_{P_l}^2 = \sum_{i=1}^{n-m} |f_i(\bar{x}_{P_l})|^2$.
Now

$$\frac{1}{K} \sum_{l=1}^{K} E_{P_l}^2 \leq \frac{\gamma^2}{4} \tag{3.6}$$

$$\Leftrightarrow \sum_{l=1}^{K} \sum_{i=1}^{n-m} |f_i(\bar{x}_{P_l})|^2 \leq \frac{K\gamma^2}{4} \tag{3.7}$$

$$\Leftrightarrow \sum_{l=1}^{K} \| \bar{x}_{P_l} \|^4 \sum_{i=1}^{n-m} |\mathcal{K}_{f_i,P_l}|^2 \leq K\gamma^2$$

We consider the following subsets of $\mathcal{N}_\varepsilon(P)$.

$$S_1 = \{P_l \in \mathcal{N}_\varepsilon(P) : \sum_{i=1}^{n-m} |\mathcal{K}_{f_i,P_l}|^2 \in \left(\sum_{i=1}^{n-m} \alpha_i^2 \ , \ \sum_{i=1}^{n-m} |\mathcal{K}_{f_i,max}|^2 \right] \},$$

$$S_2 = \{P_l \in \mathcal{N}_\varepsilon(P) : \sum_{i=1}^{n-m} |\mathcal{K}_{f_i,P_l}|^2 \in \left[\sum_{i=1}^{n-m} |\mathcal{K}_{f_i,low}|^2 \ , \ \sum_{i=1}^{n-m} \alpha_i^2 \right] \}.$$

Note: $\alpha_i \in [|\mathcal{K}_{f_i,low}| \ , \ |\mathcal{K}_{f_i,max}|] \ ; \ \forall \ i = 1, ..., n-m$. Observe that, $S_1 \cap S_2 = \phi$ and $S_1 \cup S_2 = \mathcal{N}_\varepsilon(P)$. Also observe that this is a natural generalization of the definition in Section 2.2, since the choices $n = 3$ and $m = 2$ yield the same subsets as in Section 2.2. Hence say $|S_1| = K\delta$ and $|S_2| = K(1-\delta)$, for $0 \leq \delta \leq 1$.
Now

$$\sum_{l=1}^{K} \| \bar{x}_{P_l} \|^4 \sum_{i=1}^{n-m} |\mathcal{K}_{f_i,P_l}|^2 \leq \left(\sum_{i=1}^{n-m} |\mathcal{K}_{f_i,max}|^2 \right) \sum_{P_l \in S_1} \| \bar{x}_{P_l} \|^4 + \left(\sum_{i=1}^{n-m} \alpha_i^2 \right) \sum_{P_l \in S_2} \| \bar{x}_{P_i} \|^4 .$$

Thus, similar to the $m = 2$ case, if we find conditions on the elements of S_1 and S_2 such that

$$\left(\sum_{i=1}^{n-m} |\mathcal{K}_{f_i,max}|^2\right) \sum_{P_l \in S_1} \| \bar{x}_{P_l} \|^4 \leq (K\gamma^2)\delta$$
$$\text{and} \quad \left(\sum_{i=1}^{n-m} \alpha_i^2\right) \sum_{P_l \in S_2} \| \bar{x}_{P_l} \|^4 \leq (K\gamma^2)(1-\delta),$$

then those conditions are sufficient to ensure that (3.6) is satisfied. Proceeding similarly to Section 2.2 we arrive at the following conditions on the norms of points in S_1 and S_2.

$$\| \bar{x}_{max,S_1} \| \leq \sqrt[4]{\frac{\gamma^2}{\left(\sum_{i=1}^{n-m} |\mathcal{K}_{f_i,max}|^2\right)}} \tag{3.8}$$

$$\text{is ensured if} \quad \varepsilon_{P_i,S_1} \leq \sqrt{\frac{\gamma}{\sqrt{\sum_{i=1}^{n-m} |\mathcal{K}_{f_i,max}|^2}} + \frac{\gamma^2 \sum_{i=1}^{n-m} \alpha_i^2}{4\sum_{i=1}^{n-m} |\mathcal{K}_{f_i,max}|^2}}, \tag{3.9}$$

where ε_{P_i,S_1} denotes the distance of $P_i \in S_1$ to P, in the ambient space.

$$\| \bar{x}_{max,S_2} \| \leq \sqrt[4]{\frac{\gamma^2}{\left(\sum_{i=1}^{n-m} \alpha_i^2\right)}} \tag{3.10}$$

$$\text{is ensured if} \quad \varepsilon_{P_i,S_2} \leq \sqrt{\frac{\gamma}{\sqrt{\sum_{i=1}^{n-m} \alpha_i^2}} + \frac{\gamma^2 \sum_{i=1}^{n-m} |\mathcal{K}_{f_i,low}|^2}{4\sum_{i=1}^{n-m} \alpha_i^2}}, \tag{3.11}$$

where ε_{P_i,S_2} denotes the distance of $P_i \in S_2$ to P, in the ambient space.

3.3 Performance analysis of optimal locally estimated linear subspace

In this section we seek to find conditions on the points in $\mathcal{N}_\varepsilon(P)$ such that the "angle" between the tangent space $T_P S$ and its estimation $\hat{T}_P S$ is upper bounded. $\hat{T}_P S$ is estimated from points in $\mathcal{N}_\varepsilon(P)$ and is optimal in the least squares sense. The notion of "angle" we use is as defined in [2]. Say the points in $\mathcal{N}_\varepsilon(P)$ are formed by sampling from within a ball of radius $\sqrt[4]{\frac{\eta^2}{\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|^2}}$

in $T_P S$. We show that a bound on the "angle" is guaranteed provided that η itself is upper bounded. Again with the same notation as in the previous section, assume we have K points in the neighbourhood of P.

$$\text{Let } X = \begin{bmatrix} x_1^{(1)} & \cdots & x_1^{(K)} \\ \vdots & & \vdots \\ x_m^{(1)} & \cdots & x_m^{(K)} \\ f_1(\bar{x}_1) & \cdots & f_1(\bar{x}_K) \\ \vdots & & \vdots \\ f_{n-m}(\bar{x}_1) & \cdots & f_{n-m}(\bar{x}_K) \end{bmatrix}_{n\times K}$$

The optimal m-dimensional linear subspace (in the least squares sense) passing through P will be the one spanned by the eigenvectors corresponding to the m largest eigenvalues of

$$XX^T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = U\Lambda U^T,$$

where

$$A = \begin{bmatrix} \sum_i (x_1^{(i)})^2 & \cdots & \sum_i (x_1^{(i)})(x_m^{(i)}) \\ \vdots & & \vdots \\ \sum_i (x_m^{(i)})(x_1^{(i)}) & \cdots & \sum_i (x_m^{(i)})^2 \end{bmatrix}_{m\times m},$$

$$B = \begin{bmatrix} \sum_i (x_1^{(i)} f_1(\bar{x}_i)) & \cdots & \sum_i (x_1^{(i)} f_{n-m}(\bar{x}_i)) \\ \vdots & & \vdots \\ \sum_i (x_m^{(i)} f_1(\bar{x}_i)) & \cdots & \sum_i (x_m^{(i)} f_{n-m}(\bar{x}_i)) \end{bmatrix}_{m\times (n-m)},$$

$$C = \begin{bmatrix} \sum_i (f_1(\bar{x}_i) x_1^{(i)}) & \cdots & \sum_i (f_1(\bar{x}_i) x_m^{(i)}) \\ \vdots & & \vdots \\ \sum_i (f_{n-m}(\bar{x}_i) x_1^{(i)}) & \cdots & \sum_i (f_{n-m}(\bar{x}_i) x_m^{(i)}) \end{bmatrix}_{(n-m)\times m},$$

$$\text{and} \quad D = \begin{bmatrix} \sum_i (f_1^2(\bar{x}_i)) & \cdots & \sum_i (f_1(\bar{x}_i) f_{n-m}(\bar{x}_i)) \\ \vdots & & \vdots \\ \sum_i (f_{n-m}(\bar{x}_i) f_1(\bar{x}_i)) & \cdots & \sum_i (f_{n-m}^2(\bar{x}_i)) \end{bmatrix}_{(n-m)\times (n-m)}.$$

Also $U = \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_m & \bar{u}_{m+1} & \cdots & \bar{u}_n \end{bmatrix}_{n\times n}$, $\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}_{n\times n}$

are respectively the eigenvector and eigenvalue matrices of XX^T. Assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$ i.e. rank$(XX^T) \geq m$. Say $\lambda_m > \lambda_{m+1} \geq \cdots \geq \lambda_n$. Clearly $\lambda_n \geq 0$.

Now

$$\hat{T}_P S = \text{span}\{\bar{u}_1, \ldots, \bar{u}_m\} \text{ and } \quad T_P S = \text{span}\{\bar{e}_1, \ldots, \bar{e}_m\}$$

where $\bar{e}_i$ is a $n \times 1$ vector with a 1 in the i^{th} position and 0 in the others. Let $\bar{u}_i = [u_{i,1} \ \ldots \ u_{i,n}]^T$. The angle θ between $\hat{T}_P S$ and $T_P S$ as per the definition in [2] is given by

$$\cos^2\theta \ := \ \det(M^T M)$$

where $[M^T_{i,j}] = [< \bar{u}_i, \bar{e}_j >]$; $1 \leq i, j \leq m$.

Now $\quad U^{(m)} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}_{n\times m}$ (where U_1 is $m \times m$ and U_2 is $(n-m) \times m$).

Here $U^{(m)}$ denotes the first m columns of U. Clearly $U_1^T U_1 + U_2^T U_2 = I_{m\times m}$. Let $E = [\bar{e}_1 \ \ldots \ \bar{e}_m]$.

$$\begin{aligned} \text{Thus,} \qquad M^T M &= (U^{(m)^T} E)(E^T U^{(m)}) \\ &= \begin{bmatrix} U_1^T & U_2^T \end{bmatrix} \begin{bmatrix} I_{m\times m} \\ 0_{n-m\times\ m} \end{bmatrix} \begin{bmatrix} I_{m\times m} & 0_{m\times n-m} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \\ &= U_1^T U_1 \\ &= I_{m\times m} - U_2^T U_2. \end{aligned}$$

$$\begin{aligned} \text{Now} \qquad Tr(U_2^T U_2) &= \| U_2 \|_F^2 \\ \Rightarrow \sum_{i=1}^{m} \mu_i &= \| U_2 \|_F^2 \ \ (\mu_1, \ldots, \mu_m \text{ are eigenvalues of } U_2^T U_2) \\ \Rightarrow \mu_{max} &\leq \| U_2 \|_F^2 \ \ (\mu_{max} = \max_i \mu_i; \ \mu_i \geq 0, \forall i). \end{aligned}$$

$$\begin{aligned} \text{Thus, } \cos^2\theta := \ \det(M^T M) = \det(I_{m\times m} - U_2^T U_2) &= \prod_{i=1}^{m} (1 - \mu_i) \\ &\geq (1 - \mu_{max})^m. \end{aligned}$$

$$\begin{aligned} \text{Hence if} \quad & \| U_2 \|_F < \tau < 1 \\ \text{then} \quad & \cos^2\theta > (1-\tau^2)^m \\ \text{or} \quad & |\theta| < \cos^{-1}\sqrt{(1-\tau^2)^m}. \end{aligned}$$

Let $\bar{u}_i = [u_{i,1} \ \dots \ u_{i,m} \ u_{i,m+1} \ \dots \ u_{i,n}]^T$ denote the i^{th} column vector of U. We have $\bar{u}_i = [\bar{u}_{i,1}^T \ \bar{u}_{i,2}^T]^T$, where $\bar{u}_{i,1} = [u_{i,1} \ \dots \ u_{i,m}]^T$ denotes the i^{th} column vector of U_1 and $\bar{u}_{i,2} = [u_{i,m+1} \ \dots \ u_{i,n}]^T$ denotes the i^{th} column vector of U_2. As $XX^T\bar{u}_i = \lambda_i\bar{u}_i$, we have

$$\begin{aligned} A\bar{u}_{i,1} + B\bar{u}_{i,2} &= \lambda_i\bar{u}_{i,1} \\ C\bar{u}_{i,1} + D\bar{u}_{i,2} &= \lambda_i\bar{u}_{i,2}. \end{aligned}$$

It suffices to analyze the equations corresponding to $\bar{u}_m$, i.e., the eigenvector corresponding to the m^{th} largest eigenvalue λ_m. We have the following.

$$\begin{aligned} & C\bar{u}_{m,1} + D\bar{u}_{m,2} = \lambda_m\bar{u}_{m,2} \\ & \Rightarrow \lambda_m \| \bar{u}_{m,2} \| \ \leq \| C \|_F \| \bar{u}_{m,1} \| + \| D \|_F \| \bar{u}_{m,2} \| \\ \text{or} \quad & (\lambda_m - \| D \|_F) \| \bar{u}_{m,2} \| \ \leq \| C \|_F \| \bar{u}_{m,1} \| \end{aligned}$$

We now assume that the points are sampled such that $\frac{\lambda_1}{\lambda_m} < r$, where $r \geq 1$ is fixed. We first derive upper and lower bounds for λ_m.

$$\begin{aligned} Tr(XX^T) = (\lambda_1 + \cdots + \lambda_m) + (\lambda_{m+1} + \dots \lambda_n) &\leq (\lambda_1 + \cdots + \lambda_m) + (n-m)\lambda_m \\ &\leq (m-1)\lambda_1 + (n-m+1)\lambda_m \\ &\leq c_r\lambda_m, \end{aligned}$$
$$\text{where} \quad c_r = ((n-m) + (m-1)r + 1).$$

$$\begin{aligned} \text{Also} \quad Tr(XX^T) = (\lambda_1 + \cdots + \lambda_m) + (\lambda_{m+1} + \dots \lambda_n) &\geq (\lambda_1 + \cdots + \lambda_m) \\ &\geq m\lambda_m. \end{aligned}$$

$$\therefore \quad \frac{1}{c_r}Tr(XX^T) \leq \lambda_m \leq \frac{1}{m}Tr(XX^T).$$

In order to have an upper bound on $\parallel \bar{u}_{m,2} \parallel$ we require $\lambda_m - \parallel D \parallel_F > 0$. Observe that

$$\lambda_m - \parallel D \parallel_F \geq \frac{1}{c_r} Tr(XX^T) - \parallel D \parallel_F$$

$$= \frac{1}{c_r}\left[\sum_{i=1}^{K}\left(\parallel \bar{x}_i \parallel^2 + \sum_{l=1}^{n-m} f_l^2(\bar{x}_i)\right)\right] - \parallel D \parallel_F$$

$$= \frac{1}{c_r}\left(\sum_{i=1}^{K} \parallel \bar{x}_i \parallel^2\right) + \frac{1}{4c_r}\sum_{i=1}^{K} \parallel \bar{x}_i \parallel^4 \left(\sum_{l=1}^{n-m} \mathcal{K}_{f_l,i}^2\right) - \parallel D \parallel_F .$$

We now derive an upper bound for $\parallel D \parallel_F$.

$$\begin{aligned}
\parallel D \parallel_F = & \left(\sum_{p,q=1}^{n-m}\left|\sum_{i=1}^{K} f_p(\bar{x}_i) f_q(\bar{x}_i)\right|^2\right)^{\frac{1}{2}} \\
& \leq \sum_{p,q=1}^{n-m}\left(\left|\sum_{i=1}^{K} f_p(\bar{x}_i) f_q(\bar{x}_i)\right|\right) \\
& \leq \sum_{p,q=1}^{n-m}\sum_{i=1}^{K}\left(|f_p(\bar{x}_i)||f_q(\bar{x}_i)|\right) \\
& = \sum_{i=1}^{K}\left(\sum_{l=1}^{n-m}|f_l(\bar{x}_i)|^2 + 2\sum_{(p<q)=1}^{n-m}|f_p(\bar{x}_i)||f_q(\bar{x}_i)|\right) \\
& \leq \sum_{i=1}^{K}\left(\sum_{l=1}^{n-m}|f_l(\bar{x}_i)|^2 + \sum_{(p<q)=1}^{n-m}\left(|f_p(\bar{x}_i)|^2 + |f_q(\bar{x}_i)|^2\right)\right) \\
& = (n-m)\sum_{i=1}^{K}\sum_{l=1}^{n-m}|f_l(\bar{x}_i)|^2 \\
& = \frac{(n-m)}{4}\sum_{i=1}^{K} \parallel \bar{x}_i \parallel^4 \left(\sum_{l=1}^{n-m}|\mathcal{K}_{f_l,i}|^2\right).
\end{aligned}$$

Hence, we have

$$(\lambda_m - \parallel D \parallel_F) \geq \frac{1}{c_r}\left(\sum_{i=1}^{K} \parallel \bar{x}_i \parallel^2\right) - \frac{1}{4}\sum_{i=1}^{K} \parallel \bar{x}_i \parallel^4 \left(\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,i}|^2\right)\left[(n-m) - \frac{1}{c_r}\right]$$

$$\geq \sum_{i=1}^{K} \parallel \bar{x}_i \parallel^2 \left(\frac{1}{c_r} - \frac{1}{4} \parallel \bar{x}_i \parallel^2 \left(\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|^2\right)\left(\frac{(n-m)c_r - 1}{c_r}\right)\right)$$

$$\geq \sum_{i=1}^{K} \parallel \bar{x}_i \parallel^2 \left(\frac{1}{c_r} - \frac{1}{4} \parallel \bar{x}_{\max,\mathcal{N}_\varepsilon(P)} \parallel^2 \left(\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|^2\right)\left(\frac{(n-m)c_r - 1}{c_r}\right)\right)$$

$$\geq \sum_{i=1}^{K} \parallel \bar{x}_i \parallel^2 \left(\frac{1}{c_r} - \frac{\eta}{4\sqrt{\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|^2}} \left(\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|^2\right)\left(\frac{(n-m)c_r - 1}{c_r}\right)\right)$$

$$= \sum_{i=1}^{K} \parallel \bar{x}_i \parallel^2 \left(\frac{1}{c_r} - \frac{\eta}{4}\sqrt{\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|^2}\left(\frac{(n-m)c_r - 1}{c_r}\right)\right)$$

Thus we see that the following condition on η suffices to ensure $\lambda_m - \parallel D \parallel_F > 0$.

$$\eta < \frac{4}{[(n-m)c_r - 1]\sqrt{\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|^2}}$$

So assuming η satisfies above condition we have

$$\parallel \bar{u}_{m,2} \parallel \leq \frac{\parallel C \parallel_F \parallel \bar{u}_{m,1} \parallel}{\lambda_m - \parallel D \parallel_F}$$

$$\leq \frac{\parallel C \parallel_F}{\lambda_m - \parallel D \parallel_F} \qquad (\text{Since} \quad \parallel \bar{u}_{m,1} \parallel \leq 1)$$

We now proceed to find an upper bound on $\parallel C \parallel_F$.

$$
\begin{aligned}
\parallel C \parallel_F &= \left(\sum_{p=1}^{n-m} \sum_{q=1}^{m} \left| \sum_{i=1}^{K} f_p(\bar{x}_i) x_q^{(i)} \right|^2 \right)^{\frac{1}{2}} \\
&\leq \sum_{p=1}^{n-m} \sum_{q=1}^{m} \left(\sum_{i=1}^{K} |f_p(\bar{x}_i)| \left| x_q^{(i)} \right| \right) \\
&= \frac{1}{2} \sum_{i=1}^{K} \left(\sum_{p=1}^{n-m} \sum_{q=1}^{m} \parallel \bar{x}_i \parallel^2 |\mathcal{K}_{f_p,i}| |x_q^{(i)}| \right) \\
&\leq \left(\frac{1}{2} \sum_{p=1}^{n-m} |\mathcal{K}_{f_p,max}| \right) \sum_{i=1}^{K} \parallel \bar{x}_i \parallel^2 \left(\sqrt{m} \parallel \bar{x}_i \parallel \right) \\
&\leq \left(\frac{\sqrt{m}}{2} \left(\sqrt[4]{\frac{\eta^2}{\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|^2}} \left(\sum_{p=1}^{n-m} |\mathcal{K}_{f_p,max}| \right) \right) \left(\sum_{i=1}^{K} \parallel \bar{x}_i \parallel^2 \right) \right)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \parallel \bar{u}_{m,2} \parallel &\leq \frac{\frac{\sqrt{m}}{2} c_r \sqrt[4]{\frac{\eta^2}{\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|^2}} \sum_{p=1}^{n-m} |\mathcal{K}_{f_p,max}|}{\left(1 - \left(\frac{(n-m)c_r - 1}{4} \right) \frac{\eta}{\sqrt{\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|^2}} \left(\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|^2 \right) \right)} \\
&= \frac{\left(2\sqrt{m} c_r \sum_{p=1}^{n-m} |\mathcal{K}_{f_p,max}| \right) L}{4 - ((n-m)c_r - 1)(\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|^2) L^2} \\
&= A_r, \quad \text{where } L = \sqrt[4]{\frac{\eta^2}{\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|^2}}.
\end{aligned}
$$

It is easy to verify that $\parallel \bar{u}_{i,2} \parallel \leq A_r \;\; \forall\, i = 1, \ldots, m-1$. Thus we have $\parallel U_2 \parallel_F^2 = \sum_{i=1}^{m} \parallel \bar{u}_{i,2} \parallel^2 \leq mA_r^2$. In other words, $\parallel U_2 \parallel_F \leq \sqrt{m} A_r$. We now derive an upper bound on η to ensure that $\parallel U_2 \parallel_F < \tau$ satisfied. We have the equivalence of the following conditions.

$$
\begin{aligned}
\sqrt{m}A_r &< \tau \\
\Leftrightarrow \quad \frac{\left(2mc_r \sum_{p=1}^{n-m} |\mathcal{K}_{f_p,max}|\right) L}{4-((n-m)c_r-1)(\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|^2)L^2} &< \tau \\
\Leftrightarrow \tau((n-m)c_r-1)\left(\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|^2\right) L^2 + \left(2mc_r \sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|\right) L - 4\tau &< 0
\end{aligned}
$$

$$
\Leftrightarrow \; L < \frac{\dfrac{4\tau}{mc_r \sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|}}{1+\sqrt{1+\dfrac{4\tau^2((n-m)c_r-1)\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|^2}{m^2c_r^2\left(\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|\right)^2}}}
$$

$$
= \frac{2G_{\tau,r}}{1+\sqrt{1+G_{\tau,r}^2((n-m)c_r-1)(\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|^2)}}
$$

$$
= H_{\tau,r} \quad \left(\text{where,} \quad G_{\tau,r} = \frac{2\tau}{mc_r \sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|}\right)
$$

$$
\begin{aligned}
\text{So} \quad \sqrt[4]{\frac{\eta^2}{\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|^2}} &< H_{\tau,r} \\
\Leftrightarrow \eta < H_{\tau,r}^2 \sqrt{\sum_{l=1}^{n-m} |\mathcal{K}_{f_l,max}|^2} \qquad (3.12)
\end{aligned}
$$

Thus using (3.12) we arrive at the following bounds on the norms for points in $\mathcal{N}_\varepsilon(P)$.

$$\text{As} \qquad \eta < H^2_{\tau,r}\sqrt{\sum_{l=1}^{n-m}|\mathcal{K}_{f_l,max}|^2}$$

$$\text{we have} \qquad \parallel \bar{x}_{max,\mathcal{N}_\varepsilon(P)} \parallel < H_{\tau,r}.$$

Furthermore as shown earlier in (3.5) we arrive at the following bound on the ambient space norm which ensures that the above tangent space norm bound is guaranteed.

$$\begin{aligned}
\varepsilon_{max,\mathcal{N}_\varepsilon(P)} &\leq \sqrt{\frac{\eta}{\sqrt{\sum_{i=1}^{n-m}|\mathcal{K}_{f_i,max}|^2}} + \frac{\eta^2\sum_{i=1}^{n-m}|\mathcal{K}_{f_i,low}|^2}{4\sum_{i=1}^{n-m}|\mathcal{K}_{f_i,max}|^2}} \\
&< \sqrt{H^2_{\tau,r} + \frac{H^4_{\tau,r}}{4}\sum_{i=1}^{n-m}|\mathcal{K}_{f_i,low}|^2} \\
&= H_{\tau,r}\sqrt{1 + \frac{H^2_{\tau,r}}{4}\sum_{i=1}^{n-m}|\mathcal{K}_{f_i,low}|^2}.
\end{aligned}$$

Conclusion: Let us consider that the points in $\mathcal{N}_\varepsilon(P)$ are sampled from within a ball of radius $\sqrt[4]{\dfrac{\eta^2}{\sum_{l=1}^{n-m}|\mathcal{K}_{f_l,max}|^2}}$ in T_PS, such that for any $r > 1$, $0 < \tau < 1$ the following conditions hold.

(i) $\eta < H^2_{\tau,r}\sqrt{\sum_{l=1}^{n-m}|\mathcal{K}_{f_l,max}|^2}$ where,

$$H_{\tau,r} = \frac{2G_{\tau,r}}{1 + \sqrt{1 + G^2_{\tau,r}((n-m)c_r - 1)(\sum_{l=1}^{n-m}|\mathcal{K}_{f_l,max}|^2)}}$$

$$G_{\tau,r} = \frac{2\tau}{mc_r\sum_{l=1}^{n-m}|\mathcal{K}_{f_l,max}|}$$

(ii) The rank of XX^T is at least m, and $\dfrac{\lambda_1}{\lambda_m} < r$ ($\lambda_1 \geq \lambda_2 \ldots \geq \lambda_m$ are the m largest eigenvalues of XX^T).

Then the modulus of the "angle" (as defined in [2]) between $\hat{T}_P S$ and $T_P S$ will be upper bounded by $\cos^{-1}\sqrt{(1-\tau^2)^m}$.

3.4 Random Sampling: Performance analysis of optimal locally estimated linear subspace

In the previous section we saw that an "angle" bound between $\hat{T}_P S$ and $T_P S$ is guaranteed if the points in $\mathcal{N}_\varepsilon(P)$ are sampled from within a ball in $T_P S$ of suitably small radius. In particular we saw that in order to guarantee an arbitrarily low "angle" bound, the radius of the sampling ball would be arbitrarily close to 0. In this section we show that if the points in $\mathcal{N}_\varepsilon(P)$ are formed by sampling independently and uniformly at random from the region $[-a, a]^m$ in $T_P S$, where a is such that it satisfies a fixed upper bound (independent of the angle bound parameter τ), then as the number of samples, $K \to \infty$,we have $\hat{T}_P S = T_P S$. We derive the expression for the bound on a, and also derive a corresponding lower bound on K such that if both these bounds are satisfied then it gurantees probabilistically an upper bound on the "angle" between $\hat{T}_P S$ and $T_P S$.

Let $\{x_1^{(i)}, x_2^{(i)}, \ldots, x_m^{(i)}, f_1(\bar{x}_i), \ldots, f_{n-m}(\bar{x}_i)\}_{i=1}^K$ denote K points from $\mathcal{N}_\varepsilon(P)$, where $\bar{x}_i = [x_1^{(i)} \; x_2^{(i)} \; \ldots \; x_m^{(i)}]$.

$$\text{Let } X^{(K)} = \begin{bmatrix} x_1^{(1)} & \cdots & x_1^{(K)} \\ \vdots & & \vdots \\ x_m^{(1)} & \cdots & x_m^{(K)} \\ f_1(\bar{x}_1) & \cdots & f_1(\bar{x}_K) \\ \vdots & & \vdots \\ f_{n-m}(\bar{x}_1) & \cdots & f_{n-m}(\bar{x}_K) \end{bmatrix}_{n\times K}.$$

Then

$$M^{(K)} = \frac{1}{K} XX^{T^{(K)}} = \begin{bmatrix} A^{(K)} & B^{(K)} \\ C^{(K)} & D^{(K)} \end{bmatrix},$$

where

$$A^{(K)} = \begin{bmatrix} \frac{1}{K}\sum_i (x_1^{(i)})^2 & \cdots & \frac{1}{K}\sum_i (x_1^{(i)})(x_m^{(i)}) \\ \vdots & & \vdots \\ \frac{1}{K}\sum_i (x_m^{(i)})(x_1^{(i)}) & \cdots & \frac{1}{K}\sum_i (x_m^{(i)})^2 \end{bmatrix}_{m\times m},$$

$$B^{(K)} = \begin{bmatrix} \frac{1}{K}\sum_i (x_1^{(i)} f_1(\bar{x}_i)) & \cdots & \frac{1}{K}\sum_i (x_1^{(i)} f_{n-m}(\bar{x}_i)) \\ \vdots & & \vdots \\ \frac{1}{K}\sum_i (x_m^{(i)} f_1(\bar{x}_i)) & \cdots & \frac{1}{K}\sum_i (x_m^{(i)} f_{n-m}(\bar{x}_i)) \end{bmatrix}_{m\times(n-m)},$$

$$C^{(K)} = B^{(K)^T},$$

and $$D^{(K)} = \begin{bmatrix} \frac{1}{K}\sum_i (f_1^2(\bar{x}_i)) & \cdots & \frac{1}{K}\sum_i (f_1(\bar{x}_i) f_{n-m}(\bar{x}_i)) \\ \vdots & & \vdots \\ \frac{1}{K}\sum_i (f_{n-m}(\bar{x}_i) f_1(\bar{x}_i) & \cdots & \frac{1}{K}\sum_i (f_{n-m}^2(\bar{x}_i)) \end{bmatrix}_{(n-m)\times(n-m)}.$$

We know that

$$f_l(\bar{x}) \;=\; \frac{1}{2}\sum_{j=1}^{m} < \bar{x}, \bar{v}_{l,j} >^2 \mathcal{K}_{f_l,j} \quad (\forall l = 1, \ldots, n-m).$$

We assume that the points are sampled independently and uniformly at random in $T_P S$ such that

$$x_j^{(i)} \;\sim\; U[-a,a] \text{ i.i.d } \forall i = 1, \ldots, K \quad \text{and} \quad j \;=\; 1, \ldots, m$$

where U denotes uniform distribution. Observe that each entry of $M^{(K)}$ is the sum of K i.i.d random variables. Therefore by the Strong Law of Large Numbers as $K \to \infty$ then $[M^{(K)}]_{i,j}$ converges a.s to $[M]_{i,j}$,$(1 \le i, j \le n)$, where each entry of M is the expected value of the random variable involved in the summation of the corresponding entry of $M^{(K)}$. Say

$$\mathrm{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Consider the entries of A.

$$\mathbb{E}[x_p x_q] = \begin{cases} 0 & \text{if } p \neq q \\ \frac{a^2}{3} & \text{if } p = q \end{cases}$$

and

$$-a^2 \ \le \ x_p x_q \ \le \ a^2 \quad (\forall \ p, q \ = \ 1, \ldots, m)$$

Consider the entries of B.

$$\begin{aligned}
\mathbb{E}[x_p f_q(\bar{x})] &= \mathbb{E}[x_p \frac{1}{2} \sum_{j=1}^{m} < \bar{x}, \bar{v}_{q,j} >^2 \mathcal{K}_{f_q,j}] \\
&= \frac{1}{2}\mathbb{E}[x_p \sum_{j=1}^{m} (x_1 v_{q,j,1} + \cdots + x_1 v_{q,j,m})^2 \mathcal{K}_{f_q,j}] \\
&= 0 \quad (\forall p = 1, \ldots, m \quad \text{and} \quad q = 1, \ldots, n-m).
\end{aligned}$$

This is because each term in the expansion of $x_p f_q(\bar{x})$ will have atleast one odd power of x_i. Therefore the expected value of each term is 0. Hence $B \ = \ C \ = 0$. Furthermore

$$\begin{aligned}
|x_p f_q(\bar{x})| &\le |x_p||f_q(\bar{x})| \\
&\le a.\frac{1}{2} \sum_{j=1}^{m} < \bar{x}, \bar{v}_{q,j} >^2 |\mathcal{K}_{f_q,j}| \\
&\le \frac{a}{2} |\mathcal{K}_{f,max}| \parallel \bar{x} \parallel^2 \quad (|\mathcal{K}_{f,max}| = \max_{q,j} |\mathcal{K}_{f_q,j}|) \\
&\le \frac{ma^3}{2} |\mathcal{K}_{f,max}| \quad (\parallel \bar{x} \parallel^2 \le \ ma^2).
\end{aligned}$$

$$\therefore \ \frac{-ma^3}{2} |\mathcal{K}_{f,max}| \le x_p f_q(\bar{x}) \le \frac{ma^3}{2} |\mathcal{K}_{f,max}| \quad (\forall \ p = 1, \ldots, m \text{ and } q = 1, \ldots n-m).$$

Consider the diagonal entries of D.

$$\begin{aligned}
D_{l,l} \ = \ \mathbb{E}[f_l^2(\bar{x})] &= \frac{1}{4}\mathbb{E}\left[\left(\sum_{j=1}^{m} < \bar{x}, \bar{v}_{l,j} >^2 \mathcal{K}_{f_l,j}\right)^2\right] \\
&\le \frac{1}{4} |\mathcal{K}_{f,max}|^2 (\mathbb{E}[\parallel \bar{x} \parallel^4]).
\end{aligned}$$

Now

$$\begin{aligned}
\mathbb{E}[\parallel \bar{x} \parallel^4] &= \mathbb{E}[(x_1^2 + \cdots + x_m^2)^2] \\
&= \mathbb{E}[\sum_{i=1}^{m} x_i^4 + 2\sum_{i<j} x_i^2 x_j^2] \\
&= \frac{ma^4}{5} + 2\frac{m(m-1)}{2}(\frac{a^2}{3})^2 \\
&= \frac{ma^4}{5} + m(m-1)\frac{a^4}{9} \\
&= \frac{m(5m+4)a^4}{45}.
\end{aligned}$$

Hence

$$0 \ \leq \ D_{l,l} \ \leq \ \frac{m(5m+4)a^4}{180}|\mathcal{K}_{f,max}|^2 \quad (l = 1, \ldots, n-m).$$

Furthermore

$$\begin{aligned}
f_l^2(\bar{x}) &= \frac{1}{4}(\sum_{j=1}^{m} < \bar{x}, \bar{v}_{l,j} >^2 \mathcal{K}_{f_l,j})^2 \\
&\leq \frac{1}{4}|\mathcal{K}_{f,max}|^2 \parallel \bar{x} \parallel^4 \\
&\leq \frac{m^2a^4}{4}|\mathcal{K}_{f,max}|^2.
\end{aligned}$$

Consider the off-diagonal entries of D.

$$\begin{aligned}
D_{p,q} &= \mathbb{E}[f_p(\bar{x})f_q(\bar{x})] \\
&= \frac{1}{4}\mathbb{E}[(\sum_{j=1}^{m} < \bar{x}, \bar{v}_{p,j} >^2 \mathcal{K}_{f_p,j})(\sum_{j=1}^{m} < \bar{x}, \bar{v}_{q,j} >^2 \mathcal{K}_{f_q,j})] \\
&\leq \frac{1}{4}|\mathcal{K}_{f,max}|^2 \mathbb{E}[\parallel \bar{x} \parallel^4] \\
&= \frac{m(5m+4)a^4}{180}|\mathcal{K}_{f,max}|^2 \quad (1 \ \leq \ p,q \ \leq n-m \ ; \ p \neq q).
\end{aligned}$$

Similarly we have

$$D_{p,q} \ \geq \ -\frac{m(5m+4)a^4}{180}|\mathcal{K}_{f,max}|^2 \quad (1 \ \leq \ p,q \ \leq n-m \ ; \ p \neq q).$$

Furthermore it is easily seen that

$$-\frac{m^2a^4}{4}|\mathcal{K}_{f,max}|^2 \ \leq \ f_p(\bar{x})f_q(\bar{x}) \ \leq \ \frac{m^2a^4}{4}|\mathcal{K}_{f,max}|^2.$$

Hence we have the following structure for M.

$$M = \begin{bmatrix} \frac{a^2}{3} I_{m\times m} & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & D_{(n-m)\times(n-m)} \end{bmatrix}$$

where

$$0 \ \le \ D_{i,i} \ \le \ \frac{m(5m+4)a^4}{180}|\mathcal{K}_{f,max}|^2,$$
$$-\frac{m(5m+4)a^4}{180}|\mathcal{K}_{f,max}|^2 \ \le D_{i,j} \ \le \ \frac{m(5m+4)a^4}{180}|\mathcal{K}_{f,max}|^2 \quad (i \neq j).$$

Observe that m of the eigenvectors of M are of the form: $[0, \ldots, 1, \ldots, 0]^T$ where the 1 appears at the i^{th} position for $i = 1, \ldots, m$. These eigenvectors have eigenvalues $\frac{a^2}{3}$. The span of these m eigenvectors is equal to $T_P S$.

$$\therefore \text{ If } \frac{a^2}{3} \ > \ \lambda_{max}(D) \text{ then } T_P S \ = \ \hat{T}_P(S) \text{ as } K \to \infty$$

where $\lambda_{max}(D)$ denotes the largest eigenvalue of D. Note that D is positive semidefinite. In case D is diagonal then any value of a which satisfies

$$\begin{aligned} \frac{a^2}{3} &> \frac{m(5m+4)a^4}{180}|\mathcal{K}_{f,max}|^2 \\ \text{or} \quad a^2 &< \frac{60}{m(5m+4)|\mathcal{K}_{f,max}|^2} = a^2_{\text{diag,bound}} \end{aligned}$$

ensures that as $K \to \infty$, then $\hat{T}_P S \ = \ T_P S$. In the general case where we don't make any assumption on the structure of D we have

$$\begin{aligned} \lambda_{max}(D) &< Tr(D) \\ &= \sum_{l=1}^{n-m} \mathbb{E}[f_l^2(\bar{x})] \\ &\le (n-m)\frac{a^4(5m+4)m}{180}|\mathcal{K}_{f,max}|^2. \end{aligned}$$

Thus in the general case, any value of a which satisfies

$$\frac{a^2}{3} > (n-m)\frac{m(5m+4)a^4}{180}|\mathcal{K}_{f,max}|^2$$

$$\text{or} \quad a^2 < \frac{60}{m(n-m)(5m+4)|\mathcal{K}_{f,max}|^2} = a^2_{\text{gen,bound}}$$

ensures that as $K \to \infty$, then $\hat{T}_P S = T_P S$. We would now like to find a lower bound on K (number of samples) which guarantees with a probability of at least $1-\beta$, individually for each entry of $M^{(K)}$ that it lies within an ϵ-interval of its expected value. Given this, it is then easy to verify using the union bound that the probability of the event where all entries of $M^{(K)}$ lie within an ϵ-interval of their expected value is at least $1 - \frac{n(n+1)}{2}\beta$. We proceed by using Hoeffding's Inequality. Consider the entries of A.

$$\mathbb{P}\left(\left|\frac{1}{K}\sum_{i=1}^{K} x_p^{(i)} x_q^{(i)} - 0\right| \geq \epsilon\right) \leq 2e^{\dfrac{-2K\epsilon^2}{(2a^2)^2}}$$
$$= 2e^{\dfrac{-K\epsilon^2}{2a^4}}$$

Thus,

$$2e^{\dfrac{-K\epsilon^2}{2a^4}} < \beta < 1$$
$$\Leftrightarrow K > \frac{ln\left(\dfrac{2}{\beta}\right) 2a^4}{\epsilon^2} = K^{(1)}_{bound}.$$

Consider the entries of B (or C).

$$\mathbb{P}\left(\left|\frac{1}{K}\sum_{i=1}^{K} x_p^{(i)} f_q(\bar{x}_i) - 0\right| \geq \epsilon\right) \leq 2e^{\dfrac{-2K\epsilon^2}{(ma^3|\mathcal{K}_{f,max}|)^2}}$$
$$= 2e^{\dfrac{-2K\epsilon^2}{m^2a^6|\mathcal{K}_{f,max}|^2}}$$

Thus,

$$2e^{\dfrac{-2K\epsilon^2}{m^2a^6|\mathcal{K}_{f,max}|^2}} < \beta < 1$$
$$\Leftrightarrow K > \frac{ln\left(\dfrac{2}{\beta}\right) m^2a^6|\mathcal{K}_{f,max}|^2}{2\epsilon^2} = K^{(2)}_{bound}.$$

Consider the diagonal entries of D.

$$\mathbb{P}\left(\left|\frac{1}{K}\sum_{i=1}^{K} f_l^2(\bar{x}_i) - \mathbb{E}[f_l^2(\bar{x})]\right| \geq \epsilon\right) \leq 2e^{\dfrac{-2K\epsilon^2}{\left(\dfrac{m^2a^4|\mathcal{K}_{f,max}|^2}{4}\right)^2}}$$
$$= 2e^{\dfrac{-32K\epsilon^2}{m^4a^8|\mathcal{K}_{f,max}|^4}}$$

Thus,

$$2e^{\dfrac{-32K\epsilon^2}{m^4a^8|\mathcal{K}_{f,max}|^4}} < \beta < 1$$
$$\Leftrightarrow K > \frac{ln\left(\dfrac{2}{\beta}\right) m^4a^8|\mathcal{K}_{f,max}|^4}{32\epsilon^2}.$$

Finally consider the off-diagonal entries of D.

$$\mathbb{P}\left(\left|\frac{1}{K}\sum_{i=1}^{K} f_p(\bar{x}_i)f_q(\bar{x}_i) - \mathbb{E}[f_p(\bar{x})f_q(\bar{x})]\right| \geq \epsilon\right) \leq 2e^{\dfrac{-2K\epsilon^2}{\left(\dfrac{m^2a^4|\mathcal{K}_{f,max}|^2}{2}\right)^2}}$$
$$= 2e^{\dfrac{-8K\epsilon^2}{m^4a^8|\mathcal{K}_{f,max}|^4}}$$

Thus,

$$2e^{\dfrac{-8K\epsilon^2}{m^4a^8|\mathcal{K}_{f,max}|^4}} < \beta < 1$$
$$\Leftrightarrow K > \frac{ln\left(\dfrac{2}{\beta}\right) m^4a^8|\mathcal{K}_{f,max}|^4}{8\epsilon^2}$$

Hence we essentially have the following lower bounds on K.

$$K^{(1)}_{bound} = \frac{ln\left(\frac{2}{\beta}\right) 2a^4}{\epsilon^2}$$
$$K^{(2)}_{bound} = \frac{ln\left(\frac{2}{\beta}\right) m^2 a^6 |\mathcal{K}_{f,max}|^2}{2\epsilon^2}$$
$$K^{(3)}_{bound} = \frac{ln\left(\frac{2}{\beta}\right) m^4 a^8 |\mathcal{K}_{f,max}|^4}{8\epsilon^2}$$

Let $K_{bound} = \max_{a,\epsilon}\{K^{(1)}_{bound}, K^{(2)}_{bound}, K^{(3)}_{bound}\}$. Thus we have seen that if $K > K_{bound}$ then with a probability of at least $1 - \frac{n(n+1)}{2}\beta$, no entry of $M^{(K)}$ will deviate from its expected value by more than ϵ. We saw earlier that if a is chosen to satisfy the appropriate bound ($a < a_{\text{gen,bound}}$ if no assumption is made for D, and $a < a_{\text{diag,bound}}$ if D is assumed to be diagonal) then as $K \to \infty$, $\hat{T}_P S = T_P S$. Thus assuming that a satisfies the appropriate bound, we now show that if ϵ is upper bounded and if K is larger than K_{bound} (for the chosen values of ϵ and a) then it guarantees with a probability of at least $1 - \frac{n(n+1)}{2}\beta$, an upper bound on the "angle"[2] between $T_P S$ and $\hat{T}_P S$. We analye the perturbation of M.

Let $M' = M + \triangle$ where

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad \triangle = \begin{bmatrix} \triangle_{11} & \triangle_{12} \\ \triangle_{21} & \triangle_{22} \end{bmatrix} \quad (\triangle_{12} = \triangle_{21}^T)$$

Say each entry of $\triangle_{ij}$ (i = 1,2; j = 1,2) $\in [-\epsilon, \epsilon]$. Let $\bar{u}_1, \ldots, \bar{u}_m$ be the eigenvectors of M' corresponding to the m largest eigenvalues $\lambda_1 \geq \cdots \geq \lambda_m$. We use the same notation as was defined in the previous section. We saw that $|\theta| < \cos^{-1}\sqrt{(1-\tau^2)^m}$ (where θ is the "angle" between $\hat{T}_P S$ and $T_P S$, $0 < \tau < 1$) is guaranteed if $\| U_2 \|_F < \tau$.
For any $1 \leq i \leq m$, we have the following [3].

$$\frac{a^2}{3} - n\epsilon \leq \lambda_i \leq \frac{a^2}{3} + n\epsilon$$

We now consider two cases based on the assumption made on the structure of D.

3.4.1 The case when no assumption is made for D

Note that for this case a is chosen such that $a < a_{\text{gen,bound}}$. We have the following.

$$\begin{aligned}
M'\bar{u}_i &= \lambda_i \bar{u}_i \\
\Rightarrow \triangle_{21}\bar{u}_{i,1} + (D + \triangle_{22})\bar{u}_{i,2} &= \lambda_i \bar{u}_{i,2} \\
\Rightarrow (\| D \|_F + \| \triangle_{22} \|_F) \| \bar{u}_{i,2} \| + \| \triangle_{21} \|_F &\geq \lambda_i \| \bar{u}_{i,2} \| \\
\text{or} \quad \| \triangle_{21} \|_F &\geq (\lambda_i - \| D \|_F - \| \triangle_{22} \|_F) \| \bar{u}_{i,2} \|
\end{aligned}$$

Now

$$\lambda_i - \| D \|_F - \| \triangle_{22} \|_F \geq \frac{a^2}{3} - n\epsilon - \frac{(n-m)ma^4(5m+4)|\mathcal{K}_{f,max}|^2}{180} - (n-m)\epsilon$$

So $\lambda_i - \| D \|_F - \| \triangle_{22} \|_F \geq 0$ is ensured if

$$\epsilon \leq \frac{\left(\dfrac{a^2}{3} - \dfrac{(n-m)ma^4(5m+4)|\mathcal{K}_{f,max}|^2}{180}\right)}{2n-m}$$

So assuming ϵ satisfies the above condition we have

$$\begin{aligned}
\| \bar{u}_{i,2} \| &\leq \frac{\| \triangle_{21} \|_F}{\dfrac{a^2}{3} - \dfrac{(n-m)ma^4(5m+4)|\mathcal{K}_{f,max}|^2}{180} - (2n-m)\epsilon} \\
&\leq \frac{\epsilon\sqrt{(n-m)m}}{\dfrac{a^2}{3} - \dfrac{(n-m)ma^4(5m+4)|\mathcal{K}_{f,max}|^2}{180} - (2n-m)\epsilon} \\
&= B_\epsilon.
\end{aligned}$$

$$\begin{aligned}
\text{Now} \quad \| U_2 \|_F^2 &= \sum_{i=1}^{m} \| \bar{u}_{i,2} \|^2 \\
&\leq mB_\epsilon^2 \\
\text{or} \quad \| U_2 \|_F &\leq \sqrt{m} B_\epsilon.
\end{aligned}$$

Thus $\| U_2 \|_F < \tau$ is ensured if

$$\sqrt{m} B_\epsilon < \tau$$

$$\text{or} \quad \frac{\epsilon m \sqrt{n-m}}{\frac{a^2}{3} - \frac{(n-m)ma^4(5m+4)|\mathcal{K}_{f,max}|^2}{180} - (2n-m)\epsilon} < \tau$$

$$\text{or} \quad \epsilon < \frac{\left(\frac{a^2}{3} - \frac{(n-m)ma^4(5m+4)|\mathcal{K}_{f,max}|^2}{180}\right)\tau}{m\sqrt{n-m} + (2n-m)\tau}.$$

Conclusion: Let us consider K points in $\mathcal{N}_\varepsilon(P)$ which are sampled independently and uniformly at random in $T_P S$ from the region $[-a, a]^m$

$$x_j^{(i)} \sim U[-a, a] \quad (\forall i = 1, \ldots, K \quad j = 1, \ldots, m) \quad \text{i.i.d}$$

Say for any $0 < \tau < 1$, $0 < \beta < 1$ the following conditions hold.

(i) $a < \sqrt{\frac{60}{m(n-m)(5m+4)|\mathcal{K}_{f,max}|^2}}$.

(ii) $0 < \epsilon < \frac{\left(\frac{a^2}{3} - \frac{(n-m)ma^4(5m+4)|\mathcal{K}_{f,max}|^2}{180}\right)\tau}{m\sqrt{n-m} + (2n-m)\tau}$.

(iii) For the choices of a and ϵ as in (i) and (ii),

$$K > \max\left\{ \frac{ln\left(\frac{2}{\beta}\right) 2a^4}{\epsilon^2}, \frac{ln\left(\frac{2}{\beta}\right) m^2 a^6 |\mathcal{K}_{f,max}|^2}{2\epsilon^2}, \frac{ln\left(\frac{2}{\beta}\right) m^4 a^8 |\mathcal{K}_{f,max}|^4}{8\epsilon^2} \right\}.$$

Then we have with probability at least $1 - \frac{n(n+1)}{2}\beta$ that $|\theta| < \cos^{-1}\sqrt{(1-\tau^2)^m}$.

3.4.2 The case when D is diagonal

Note that for this case a is chosen such that $a < a_{\text{diag,bound}}$. We have the following.

$$
\begin{aligned}
M'\bar{u}_i &= \lambda_i \bar{u}_i \\
\Rightarrow \triangle_{21}\bar{u}_{i,1} + (D + \triangle_{22})\bar{u}_{i,2} &= \lambda_i \bar{u}_{i,2} \\
\Rightarrow \sum_{j=1}^{m}([\triangle_{21}]_{p,j}u_{i,1}^{(j)}) + (D_{p,p} + [\triangle_{22}]_{p,p})u_{i,2}^{(p)} + \sum_{j=1, j\neq p}^{n-m}[\triangle_{22}]_{p,j}u_{i,2}^{(j)} &= \lambda_i u_{i,2}^{(p)} \quad (p = 1, \ldots, n-m) \\
\Rightarrow (\lambda_i - D_{p,p} - \epsilon)|u_{i,2}^{(p)}| &\leq m\epsilon + (n-m-1)\epsilon \\
&= (n-1)\epsilon.
\end{aligned}
$$

Now

$$
\begin{aligned}
\lambda_i - D_{p,p} - \epsilon &\geq \lambda_i - \frac{m(5m+4)a^4|\mathcal{K}_{f,max}|^2}{180} - \epsilon \\
&\geq \frac{a^2}{3} - n\epsilon - \frac{m(5m+4)a^4|\mathcal{K}_{f,max}|^2}{180} - \epsilon \\
&= \frac{a^2}{3} - \frac{m(5m+4)a^4|\mathcal{K}_{f,max}|^2}{180} - (n+1)\epsilon.
\end{aligned}
$$

Therefore $\lambda_i - D_{p,p} - \epsilon \geq 0$ is ensured if

$$
\epsilon < \frac{\frac{a^2}{3} - \frac{m(5m+4)a^4|\mathcal{K}_{f,max}|^2}{180}}{n+1}.
$$

Thus assuming ϵ satisfies the above condition we have the following

$$
|u_{i,2}^{(p)}| \leq \frac{(n-1)\epsilon}{\frac{a^2}{3} - \frac{m(5m+4)a^4|\mathcal{K}_{f,max}|^2}{180} - (n+1)\epsilon} = B_\epsilon.
$$

The above holds $\forall\ i = 1, \ldots, m$ and $p = 1, \ldots, n-m$. We have the following upper bound on $\parallel U_2 \parallel_F$.

$$
\begin{aligned}
\parallel U_2 \parallel_F^2 &= \sum_{i=1}^{m}\sum_{p=1}^{n-m}(u_{i,2}^{(p)})^2 \\
&\leq m(n-m)B_\epsilon^2 \\
\text{or} \quad \parallel U_2 \parallel_F &\leq \sqrt{m(n-m)}B_\epsilon.
\end{aligned}
$$

Thus the following suffices to ensure that $\parallel U_2 \parallel_F < \tau$.

$$\sqrt{m(n-m)}B_\epsilon < \tau$$

$$\text{or} \quad \frac{\sqrt{m(n-m)}(n-1)\epsilon}{\frac{a^2}{3} - \frac{m(5m+4)a^4|\mathcal{K}_{f,max}|^2}{180} - (n+1)\epsilon} < \tau$$

$$\text{or} \quad \epsilon < \frac{\left(\frac{a^2}{3} - \frac{m(5m+4)a^4|\mathcal{K}_{f,max}|^2}{180}\right)\tau}{\sqrt{m(n-m)}(n-1) + (n+1)\tau}.$$

Conclusion: Let us consider K points in $\mathcal{N}_\varepsilon(P)$ which are sampled independently and uniformly at random in $T_P S$ from the region $[-a,a]^m$

$$x_j^{(i)} \sim U[-a,a] \quad (\forall i = 1,\ldots,K \quad j = 1,\ldots,m) \quad \text{i.i.d}$$

Say for any $0 < \tau < 1$, $0 < \beta < 1$ the following conditions hold.

(i) $a < \sqrt{\dfrac{60}{m(5m+4)|\mathcal{K}_{f,max}|^2}}$.

(ii) $0 < \epsilon < \dfrac{\left(\frac{a^2}{3} - \frac{m(5m+4)a^4|\mathcal{K}_{f,max}|^2}{180}\right)\tau}{\sqrt{m(n-m)}(n-1) + (n+1)\tau}$.

(iii) For the choices of a and ϵ as in (i) and (ii),

$$K > \max\left\{ \frac{ln\left(\frac{2}{\beta}\right)2a^4}{\epsilon^2}, \frac{ln\left(\frac{2}{\beta}\right)m^2a^6|\mathcal{K}_{f,max}|^2}{2\epsilon^2}, \frac{ln\left(\frac{2}{\beta}\right)m^4a^8|\mathcal{K}_{f,max}|^4}{8\epsilon^2} \right\}.$$

Then we have with probability at least $1 - \frac{n(n+1)}{2}\beta$ that $|\theta| < \cos^{-1}\sqrt{(1-\tau^2)^m}$.

Chapter 4

Conclusions

In this work we did a theoretical analysis of the local sampling conditions for a set of discrete points lying on a quadratic embedding of a Riemannian manifold in a Euclidean space. Our analysis was based on the following criteria: (i) Local reconstruction error (ii) Local tangent space estimation accuracy. We first did the analysis for quadratic surfaces embedded in $\mathbb{R}^3$ and then extended it for the general case of quadratic embeddings of m-dimensional Riemannian manifolds in $\mathbb{R}^n$.

In the local reconstruction error analysis we described sampling conditions in the neighbourhood of a reference point P on the manifold such that the average reconstruction error of the neighbourhood points after orthogonal projection on the tangent space at P satisfies a given upper bound. We analyzed the local neighbourhood, $\mathcal{N}_\varepsilon(P)$ by considering disjoint regions, namely the local "high" curvature (S_1) and "low" curvature regions (S_2) respectively. For the case of quadratic surfaces embedded in $\mathbb{R}^3$, we showed that the sampling regions which guarantee a reconstruction error criterion geometrically correspond to disjoint sectors of two concentric discs in the tangent space at P, where each one of these discs is associated to S_1 or S_2. In particular we showed precisely that the points lying in S_1 could lie closer to P as compared to the points in S_2, in order to satisfy a local reconstruction error criterion. We also considered the case where the coordinates of the points in the neighbourhood are sampled uniformly at random from scaled sampling regions in the tangent space at P. We derived a condition for the scaling factors of the norms of points in S_1 and S_2. It was shown that if the scaling factors satisfy this condition and if the number of samples K satisfies a lower bound then it guarantees an upper bound on the probability of the event where the empirical average reconstruction error of the points in $\mathcal{N}_\varepsilon(P)$ exceeds the reconstruction error criterion. We validated our theoretical results by running simulations on synthetic quadratic surfaces with different principal curvature values at P. Lastly we derived analogous theoretical results for the case of quadratic embeddings of m-dimensional Riemannian manifolds in $\mathbb{R}^n$.

In local tangent space estimation analysis we derived conditions for points in $\mathcal{N}_\varepsilon(P)$ so that the "angle"[2] between the original tangent space and the estimated tangent space at P satisfies a given upper bound. The tangent space was estimated using the points in the neighbourhood of P. We showed two possibilities to bound the angle. In the first case we showed precisely how close the points should be to P in order to satisfy an angle bound. In particular this distance to P depends on the angle bound parameter in the sense that as the angle bound becomes arbitrarily small then the distance of the points from P should also be suitably small. In the second case we considered the coordinates of the points to be sampled uniformly and independently at random from a region in the tangent space at P. For quadratic surfaces embedded in $\mathbb{R}^3$ we considered the sampling region to be a disc in the tangent space. We showed that if the radius of the disc satisfies an upper bound and if the number of samples in the region satisfies a lower bound then it guarantees probabilistically an upper bound on the "angle" between the estimated tangent space and the original tangent space. In particular the upper bound on the radius of the disc depends solely on the principal curvature values of the surface at P, and is independent of the angle bound parameter. For quadratic embeddings of m-dimensional Riemannian manifolds in $\mathbb{R}^n$ we considered the sampling region to be of the form $[-a, a]^m$. We again showed that if a satisfies an upper bound and if the number of samples in the region satisfies a lower bound then it guarantees probabilistically an upper bound on the "angle" between the estimated tangent space and the original tangent space.

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